PART IV: Cinematography & Mathematics

AGE RANGE: 16-18

TOOL 36: PROBABILITIES IN "21"

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Educator's Guide

Title: Probabilities in the movie "21" by Robert Luketic

Age Range: 16-18 years old

Duration: 2 hours

Mathematical Concepts: Fibonacci, factorials, permutations, combinations, Pascal's Triangle, probability

'Artistic' Concepts: Blackjack, card counting

General Objectives: To discover the mathematical concepts presented in the movie and acquire a more practical view of the use of math in a commonly played game.

Instructions and Methodologies: The students will explore math by playing cards themselves and watching the suggested videos. This tool will help your class discover the different math concepts needed to learn probability.

Resources: This tool provides pictures and videos for you to use in your classroom. The topics addressed in these resources will also help you find other materials to personalize and give nuance to your lesson.

Tips for the educator: Learning by doing is very efficient, especially for learners with learning difficulties. Always explain the practical use of each math concept.

Desirable Outcomes and Competences: At the end of this tool, the student will be able to:

- Understand Factorials;
- Use Permutations and Combinations;
- Calculate probabilities.

Debriefing and Evaluation:

Write 3 aspects you liked about this activity:	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

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Introduction

Watching a movie can either be an active or a passive leisure activity. Movies can be valuable resources for learners to explore the different topics addressed. Some of them use mathematics in their plots, which students often don't really focus on though they will be more likely to understand a topic they heard about in a movie.

Seeing the characters reflect on mathematical problems and concepts makes the viewer want to understand those concepts and solve those problems with them in the same way as they often try to guess the end of a movie, here they will learn new things just by following characters throughout the story.

Therefore, teaching students the mathematics that hide behind the movies can be a great added value to a math course, often considered too abstract, by giving learners a more practical and real-life sense of the possible uses of mathematics.

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“21” by Robert Luketic

Synopsis

Figure 1: Poster of the movie "21"

The movie « 21 » is about a group of students at the MIT who decide to use card counting to win more Blackjack games in casinos. This method is very well-known and is based on probabilities.

Movie Trailer:
https://www.youtube.com/watch?v=oqkdB7It5Go

Blackjack

As you know, this movie is about Blackjack, but did you know that Blackjack was originally a French game in the 1760s called “Vingt-et-un”, 21 in French, just as the movie’s title!

In this movie, Ben is very good at math and his professor asks him to join his card counting club to play in Las Vegas. Ben needs money to pay for university and agrees to play until he earns the amount of money he needs. This movie was based on real facts, there really was an MIT team who trained to count cards and play Blackjack.

Learn more in this documentary:
https://www.youtube.com/watch?v=QflVqavHHM0

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Glossary

MIT: Massachusetts Institute of Technology, a very a very well-known private university in the US.

Blackjack: a card game played with 52 cards. It is commonly played at the casino.

Card Counting: a technique used in casino games such as Blackjack to calculate the odds of winning.

Vingt-et-un: a French gambling card game from the 18th century that preceded Blackjack.

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The math behind 21

Introduction activity: Fibonacci's Sequence:

The first math concept that appears in "21" is the Fibonacci Sequence on Ben's birthday cake. As he studies mathematics at MIT, his friends have made him a special cake for his birthday. The Fibonacci sequence is one in which you start with the numbers 0 and 1 and each following number is the sum of the two former ones. It goes like this: 0 + 1 = 1; 1 + 1 = 2 ; 1 + 2 = 3 ; 2 + 3 = 5, and so on.

Figure 2: Screenshot of the birthday scene in "21"

The numbers on Ben's cake are 0,1,1,2,3,5,8,13… Why did Ben's friends stop there? Can you guess Ben's age?

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1. Factorials

Luck games don't always depend so much on luck. You can use something called probability to guess your chances to win the game.

Before you learn how to calculate probabilities, let's start with a short video on how 52 cards can be arranged:

https://www.youtube.com/watch?v=uNS1QvDzCVw.

Did you expect that? Now we understand why these students needed to train so much before going to Las Vegas!

As you saw in the video, the number of arrangements possible for a group of elements is called "Factorials".

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Remember this formula:

The factorial **n!** is equal to:

n x (n-1) x (n-2) x (n-3) … 3 x 2 x 1

or

n! = n x (n-1)!

If you want to calculate a fraction of factorials, don't forget to simplify it as follows:

$$\frac{10!}{6!} = \frac{10x9x8x7x6x5x4x3x2x1}{6x5x4x3x2x1} = \frac{10x9x8x7x\cancel{6!}}{\cancel{6!}} = 10 \times 9 \times 8 \times 7 = 5040$$

Remember: it is generally accepted that 0! = 1

2. Permutations

The permutation of a set of elements is an ordered arrangement of all the elements of that set.

A Permutation, written $_nP_r$, ^nP_r, or P(n,r), can either allow repetitions or not:

- **Permutations with Repetition**: these are easy to calculate as there are n, the number of things to choose from and r, the choices.

  The formula is: **n^r = n x n x n… (r times)**

Example:

If we need to find a three-digit code to lock out phone where there can be repetitions, we have 10 numbers to choose from and 3 choices to make.

n = 10 and r = 3

$^{10}P_3 = 10^3 = 1000$

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- **Permutations without Repetition**: the difference is that we reduce the number of choices. To avoid repetitions, the formula is no longer n x n x n x … but becomes **n x (n-1) x (n-2) x (n-3) … = n!** the factorial. However, if we only want to choose r of them, we can reduce the formula to $^{n}P_{r} = \frac{n!}{(n-r)!}$.

Example:

We are playing pool and have 16 pool balls. Each of them appears once, so there are no repetitions. There are 16 x 15 x 14 x 13 x 12 x 11… x 2 x 1 permutations.

If we want to choose 5 of them, the calculation will be different:

$$^{16}P_{5} = \frac{16!}{(16-5)!} = \frac{16!}{11!} = \frac{16 \times 15 \times 14 \times 13 \times 12 \times \cancel{11!}}{\cancel{11!}} = 16 \times 15 \times 14 \times 13 \times 12 = 524\ 160$$

3. Combinations:

The combination of a set of elements is a non-ordered arrangement of all elements of that set.

- **Combinations Without Repetition**:

Say we want to play Blackjack and ten people will sit in five chairs around the table. If we try to find the number of permutations there is, we would use the formula

$^{n}P_{r} = \frac{n!}{(n-r)!}$.

The calculation would be:

$$^{10}P_{5} = \frac{10!}{(10-5)!} = \frac{10!}{5!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!} = 10 \times 9 \times 8 \times 7 \times 6 = 30\ 240$$

In this case, we don't care about the order, so we need to divide that number into the number of ways there is to arrange 5 people:

$$^{n}C_{r} = \frac{n!}{(n-r)!} \times \frac{1}{r!} = \frac{n!}{r!(n-r)!}$$

Let's apply that formula to our case:

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$$^{10}C_5 = \frac{10!}{5!(10-5)!} = \frac{10!}{5! \times 5!} = \frac{10!}{120 \times 120} = \frac{3\,628\,800}{14\,400} = 252$$

You can also use Pascal's triangle, which is treasured by mathematicians since it was discovered.

You can learn more about it in the following video: https://www.youtube.com/watch?v=XMriWTvPXHI&t=80s

You just need to go down to the n^{th} row and pick the r^{th} number (the first row and number are the lines 0).

Let's try that with this example:

You're at a blackjack game and you need to sit 4 people in 3 chairs. According to Pascal's triangle, there will be 4 combinations.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Let's check with the formula:

$$^{4}C_3 = \frac{4!}{3!(4-3)!} = \frac{4!}{3!1!} = \frac{24}{6 \times 1} = \mathbf{4}$$

Here's another example:

If we have a 52-card game which we want to divide into hands of 2 cards to play Blackjack, how many combinations are there of those two cards?

$$^{52}C_2 = \frac{52!}{2!(52-2)!} = \frac{52!}{2!50!} = \frac{52 \times 51}{2!} = 1\,326 \text{ combinations}$$

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❖ **Combinations With repetitions:** Here, the order doesn't matter but you can repeat several times the same choice.

Ben's friends need to choose 4 flavors for Ben's birthday cake among the 7 flavors offered. Let's call them A, B, C, D, E, F, and G.
How many combinations with repetitions will we have?

A, B, C, D, E, F, G; where:

- V are the chosen ingredients and
- → are the movements (from A to B, B to C, C to D, etc.)

A A D F	V V → → → V → → V →
C E E G	→ → V → → V V → → V
B C F F	→ V → V → → → V V →

…

You can notice there are always 6 → and 4 V. This brings us to the transformation of the formula for combinations without repetitions and gives us the formula for combinations with repetitions:

$$^{n}C_{r} = \frac{(r+n-1)!}{r!(n-1)!}$$

Let's apply this formula to our problem:

$$^{7}C_{4} = \frac{(4+7-1)!}{4!(7-1)!} = \frac{10!}{4!6!} = \frac{10!}{24 \text{ x } 720} = \frac{3\,628\,800}{17\,280} = 210$$

4. Probability

Probability is used to get a more precise idea of the chances that something random has to happen.

It is written P(A) and will always be between 0 and 1.
Remember: $\mathbf{0 \leq P(A) \leq 1}$
As two complementary events make up all the possibilities: $\mathbf{P(A_C) + P(A) = 1}$

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To calculate it, we divide the number of possibilities for our choice by the number of equally likely possibilities:

$$\frac{\#\text{ of possibilities for our choice}}{\#\text{ of equally likely possibilities}}$$

This means that, for instance, when you roll a dice, you have $\frac{1}{6}$ possibilities to have a 4 and $\frac{2}{6}$ possibilities to have a 4 or a 5. It is however impossible to have both a 4 and a 5, so it is $\frac{0}{6}$.

Example: The Monty Hall Problem

In one of his courses, Ben's professor states a mathematical problem where:

- There are three doors in front of you;
- Behind one of the doors, there is a new car
- Behind the other two there are goats.
- You want to choose the one with the new car but there is $\frac{1}{3}$ chance to get the car and $\frac{2}{3}$ chances to get a goat.

Ben chooses door 1 and his professor shows him that behind door number 3, there is a goat. He offers to change his answer. Now that he knows where one of the goats is, should he take this second chance?

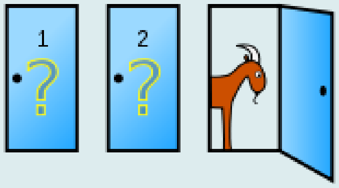

Figure 3: Visual representation of the Monty Hall Problem

You can see the answer here:
https://www.youtube.com/watch?v=8DMnAAvakh0

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Here's another example:

You have 12 cards in your hands and your neighbor has to pick one. In those 12 cards, you have three tens, five Kings, two Aces and two fours. What is the chance that your neighbor picks a King?

The possible outcomes are: {10, 10, 10, K, K, K, K, K, A, A, 4, 4}

So, the probability of getting a King, P(K)= $\frac{5}{12}$

Before we go any further, you need to know some formulas for these particular calculations:

To calculate the probability of two events occurring at the same time, multiply their probabilities.

P(A∩B) = P(A) x P(B)

In a 52-card game, what are the possibilities to pick a Queen and a Jack?

P(J∩Q) = P(J) x P(Q)

P(J∩Q) = $\frac{1}{13} \mathbf{x} \frac{1}{13} = \frac{1}{169}$ = 0,006

To calculate the probability of either one of two events happening, add their probabilities then take out the probability of both happening at the same time.

P(A∪B) = P(A) + P(B) – P(A∩B)

For example, in a 52-card game, what are the possibilities to pick a Queen or a Jack?

P(Q∪J) = P(Q) + P(J) – P(Q∩J)

P(Q∪J) = $\frac{1}{13}+\frac{1}{13}-\frac{1}{169}=\frac{13}{169}+\frac{13}{169}-\frac{1}{169}=\frac{25}{169}=\mathbf{0,148}$

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TASK

Let's play Blackjack:

We play Blackjack with 52 cards; each card has its own value. The aim of the game is to have a sum of 21 points. An Ace and a 10-point card are the best hand. The Ace will turn into a 1 when the sum is more than 21. All players get two cards face up. The dealer gets one card face up and one face down. If the dealer has Blackjack (21 points), he reveals his cards and wins the bet along with the players who also have Blackjack. If the dealer loses, it is the other players' turn to either keep their cards, take one more, split in two different hands, or surrender and get half their bet.

Here are the values of all the cards:

Ace	2	3	4	5	6	7	8	9	10	Jack	Queen	King
1 or 11	2	3	4	5	6	7	8	9	10	10	10	10

Figure 4: Picture of a casino cards game

We have five players around the table. They each have 2 cards. Among those cards, there are four Aces, two tens, one Jack, one four and two fives. What are the chances to get two cards that add up to 21?

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LEARN MORE…

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Pascal's Triangle:

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Probabilities:

https://www.youtube.com/watch?v=3V2omKRX9gc

Probabilities:

https://www.youtube.com/watch?v=Kgudt4PXs28

Ben's Answer to the Monty Hall Problem:

https://www.youtube.com/watch?v=8DMnAAvakh0

Documentary about MIT Team:

https://www.youtube.com/watch?v=QflVqavHHM0

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