PART V: Literature & Mathematics

AGE RANGE: 16-18

TOOL 49: CONIC SECTIONS IN "ALICE'S ADVENTURES IN WONDERLAND" BY LEWIS CARROLL

LogoPsyCom

TheArtOf Maths

Co-funded by the Erasmus+ Programme of the European Union

Educator's Guide

Title: Mathematics in "Alice's Adventures in Wonderland" by Lewis Carroll (1865)

Age Range: 16-18 years old

Duration: 3 hours

Mathematical Concepts: Euclidean and non-Euclidean geometries, conic sections.

Artistic Concepts: Literary analysis, the novel, metaphors.

General Objectives: To discover the mathematical concepts presented in the book and learn how to build math reasoning in everyday life.

Instructions and Methodologies: The students will explore math through literature, by applying it to real-life situations and reading the excerpts. Your class will discover different math concepts to learn about conic sections.

Resources: This tool provides pictures and videos. The topics addressed in these resources will help you find other materials to personalize and nuance to your lesson.

Tips for the educator: Learning by doing is very efficient, especially for young learners with learning difficulties. Always explain the practical use of each math concept.

Desirable Outcomes and Competences: At the end of this tool, the student will be able to:

- Understand the difference between Euclidean and non-Euclidean geometry;
- Understand what conic sections are;
- Use a quadratic equation to draw the graph of a function.

Debriefing and Evaluation:

Write 3 aspects you liked about this activity:	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

Introduction

Reading can help us understand the world around us in a way we didn't expect. Books are thus valuable resources for learners to explore new topics and concepts hidden within the story. Some of the authors use mathematics in their plots, which students often don't really focus on though they will be more likely to understand a topic they have already read about.

Seeing the characters reflect on mathematical problems and concepts makes the reader want to understand those concepts and solve those problems with them in the same way as readers often try to guess the end of a story. Here, they will learn new things just by following the characters' adventures.

Therefore, teaching students the mathematics that hide behind some well-known books can be a great added value to a math course, by giving learners a more immersive experience of the possible uses of mathematics.

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

"Alice's adventures in Wonderland"

By Lewis Carroll

1. Synopsis

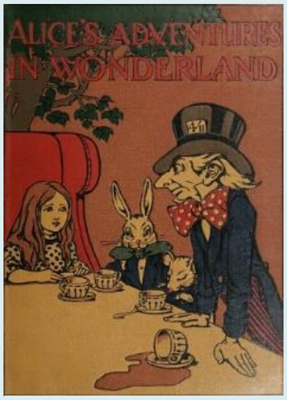


Figure 1: Cover page of "Alice's Adventures In Wonderland"

This novel written by Lewis Carroll in 1865 tells the story of a seven-year-old girl, Alice, who falls into a rabbit hole and ends up in Wonderland, an imaginary place with peculiar people and talking animals. She goes from place to place, changing size and shape and lives unexpected adventures. You might be more familiar with its Disney adaptation into an animated movie.

2. The Context

Before we go deeper in the story, it is important to know about the context in which it was written. The author's name was Charles Lutwidge Dodgson (he used Lewis Carroll as a pseudonym). He was a math tutor at the Christ Church College in Oxford and was a very conservative mathematician. He based all his mathematical knowledge on Euclid's "Elements". During the 19th century, mathematics were changing and new theories were rising. For instance, abstract algebra and "imaginary numbers" emmerged at that time. In order to show the absurdity of these new mathematics, he used methaphors throughout Alice's story.

3. The Caterpillar scene

In this scene, Alice has been changing size several times by eating magic mushrooms. She encounters a caterpillar who is smoking a hookah. The word "algebra" comes from the Arabic "al jebr e al mokabala" which means "restoration and reduction". This is exactly what Alice does when she eats a mushroom, hoping to restore her original size,

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

and ends up shrinking from almost 3m to 8cm. This scene sets the tone for the following adventures as Alice will need to eat the exact amount of mushroom necessary to find balance and keep her body's size and proportions.

4. The Pig and Pepper scene

In this scene, Alice is in the Duchess' kitchen and everyone starts to sneeze due to the cook adding too much pepper to the preparation. The duchess hands her baby over to Alice. The baby then slowly turns into a pig, which Alice only realizes when she hears the baby grunting instead of sneezing. This scene shows the absurdity of projective geometry, which studies if a shape's properties can stay the same when it is projected onto another surface, provided that it keeps its basic properties. The author employs a Euclidean technique called "reductio ad absurdum" to say that "if the rule works for a triangle, it should work for a baby too. QED". The baby keeps some of his basic properties, still being pink and pudgy, which is why Alice only realizes the metamorphosis when he grunts.

EXCERPT:

"the Duchess was sitting on a three-legged stool in the middle, nursing a baby; the cook was leaning over the fire, stirring a large cauldron which seemed to be full of soup.
'There's certainly too much pepper in that soup!' Alice said to herself, as well as she could for sneezing. There was certainly too much of it in the air. Even the Duchess sneezed occasionally; and as for the baby, it was sneezing and howling alternately without a moment's pause. The only things in the kitchen that did not sneeze, were the cook, and a large cat which was sitting on the hearth and grinning from ear to ear.
'Please would you tell me,' said Alice, a little timidly, for she was not quite sure whether it was good manners for her to speak first, 'why your cat grins like that?'
'It's a Cheshire cat,' said the Duchess, 'and that's why. Pig!' She said the last word with such sudden violence that Alice quite jumped; but she saw in another moment that it was addressed to the baby, and not to her […]

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

And with that she began nursing her child again, singing a sort of lullaby to it as she did so, and giving it a violent shake at the end of every line:

'Speak roughly to your little boy,
And beat him when he sneezes:
He only does it to annoy,
Because he knows it teases.'
CHORUS.
(In which the cook and the baby joined):
'Wow! wow! wow!' [...]

'Here! you may nurse it a bit, if you like!' the Duchess said to Alice, flinging the baby at her as she spoke. 'I must go and get ready to play croquet with the Queen,' and she hurried out of the room. The cook threw a frying-pan after her as she went out, but it just missed her.

Alice caught the baby with some difficulty, as it was a queer-shaped little creature, and held out its arms and legs in all directions, 'just like a star-fish,' thought Alice. The poor little thing was snorting like a steam-engine when she caught it, and kept doubling itself up and straightening itself out again, so that altogether, for the first minute or two, it was as much as she could do to hold it. [...]

'If I don't take this child away with me,' thought Alice, 'they're sure to kill it in a day or two: wouldn't it be murder to leave it behind?' She said the last words out loud, and the little thing grunted in reply (it had left off sneezing by this time). 'Don't grunt,' said Alice; 'that's not at all a proper way of expressing yourself.' The baby grunted again, and Alice looked very anxiously into its face to see what was the matter with it. There could be no doubt that it had a *very* turn-up nose, much more like a snout than a real nose; also its eyes were getting extremely small for a baby: altogether Alice did not like the look of the thing

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

at all. 'But perhaps it was only sobbing,' she thought, and looked into its eyes again, to see if there were any tears.

No, there were no tears. 'If you're going to turn into a pig, my dear,' said Alice, seriously, 'I'll have nothing more to do with you. Mind now!' The poor little thing sobbed again (or grunted, it was impossible to say which), and they went on for some while in silence."

---

5. The Tea Party scene

The tea party at the Mad Hatter's tries to demonstrate the absurdity of mathematician William Rowan Hamilton's works, in which he experimented with quartenions, which is a system of numbers based on four terms. The three first terms stand for the three dimensions and the fourth is an extra-spatial unit, the concept of Time.

The title of this chapter "tea-party" which can be read as 't-party", since "t" is the mathematical symbol for time. The three guests of the tea party, the Mad Hatter, the Dormouse, and the March Hare represent the three initial terms. Time remains the fourth and the Mad Hatter says they had a quarrel and now Time won't do a thing he asks. As a result, all three guests are now forced to turn on a plane around the table for eternity. The Hatter's riddle "Why is a raven like a writing desk?" doesn't make any sense, which can also reflect that in Hamilton's idea of pure time, the cause and its effect are no longer linked. The author used his creativity to demonstrate the absurdity of the new mathematical theories, which created a nonsensical world.

To learn more about the history of Time, you can watch this TED-Ed video: https://www.youtube.com/watch?v=R3tbVHlsKhs.

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

Glossary

Pseudonym: a false name used by an artist, author, etc. to sign their work.

Euclid (4th – 3rd century BC): a Greek mathematician from the Antiquity who established the grounds of geometry.

Abstract Algebra: a branch of algebra that studies algebraic structures.

Imaginary Numbers: are numbers that can be written as real numbers multiplied by the imaginary unit i, of which we know that $\boldsymbol{i} = -\mathbf{1}$. For instance, ai=-$a^2$. In this theory, zero is both imaginary and real.

Methaphor: it is a figure of speech that allows us to refer to something implicitly by talking about something different but showing the similarity we want to emphasize. Ex: to be fishing for compliments : the person is not literally fishing, but we can use this image as a methaphor.

Hookah: an oriental pipe which has a long tube that draws smoke from a bowl filled with water.

Reductio ad absurdum (latin): means "reduction to absurdity" and is an argument used in logic to show that a statement is ridiculous and cannot be taken seriously.

QED (latin): "Quod erat demonstrandum" means "what was to be shown" and is often used at the end of mathematical or philosophical demonstrations.

Quartenions: is a number system which extends the complex numbers. One of their features is that the multiplication of two quartenions is not commutative.

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

The math behind Alice's adventures in Wonderland

You are going to learn about Euclidean and non-Euclidean geometries, which will allow you to understand what the author agreed with and what he found completely absurd.

Euclidean and non-Euclidean Geometries

1. Euclidean Geometry

Euclidean geometry is the one we learn about the most. Euclid was the "father of geometry" and wrote his postulates, definitions and common notions in his books "Elements". The fifth postulate brought out some reflections among some other renowned mathematicians throughout history. Euclid based his geometry on distance and angles, which will not be the case in the following studies on geometry.

Watch the following TED-Ed video about Euclidean and non-Euclidean geometry: https://www.youtube.com/watch?v=LPET_HhN0VM

2. Non-Euclidean Projective Geometry:

Projective geometry is used in non-Euclidean geometries such as Elliptic and Hyperbolic geometries. It focuses on the projection of shapes onto other surfaces and states that **a shape can bend or stretch into another one if it keeps its basic properties**. It assumes that Euclid's fifth postulate is wrong.

The curvature of the surface on which the shapes are projected is a big part of what makes non-Euclidean geometries different from Euclidean geometry. Perspective, for instance, is one of the results of this reasoning. Mathematicians studied the properties of projected shapes in order to see if they remained the same with perspective.

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

Example of two parallel lines:

Figure 2: Image of a road seen from the viewer's perspective

According to Euclidean geometry, two parallel lines should never meet but when looked at with perspective or projected onto another surface, those lines seem to meet in the horizon, at some point in **infinity**. With the concept of infinity, we are faced with a geometry that does not consider the shapes' angles and distances.

Conic Sections

When you cut a cone with planes in different parts, you will get a different shape. Let's have a look at the following image:

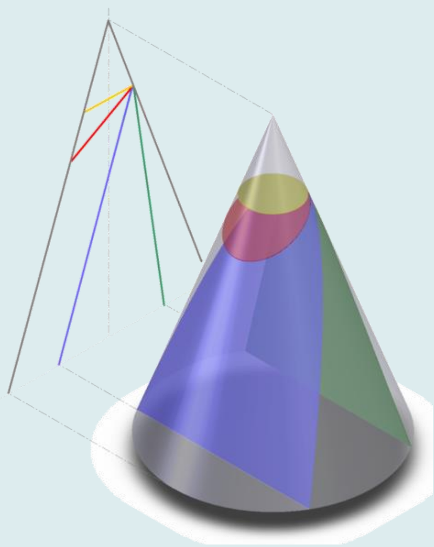


Figure 3: Representation of a conic section

As you can see, as we cut the cone with a plane, we will have different shapes.

- In Yellow: we cut straight through it horizontally and end up with a circle.
- In **Red**: we cut it with a slight angle, and we get an **ellipse**.
- In **Blue**: we cut into it diagonally and parallel to the cone's edge and get a **parabola**.
- In **green** we cut into it on one side with a steep angle and get a **hyperbola**.

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

You can also use a point (Focus) and a straight line (directrix) to define these curves:

Measure the distance:

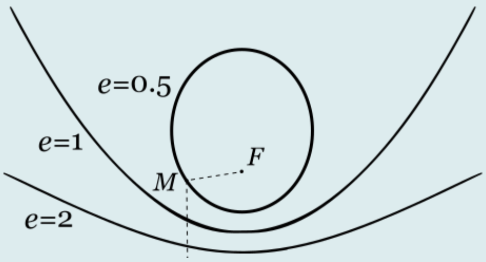


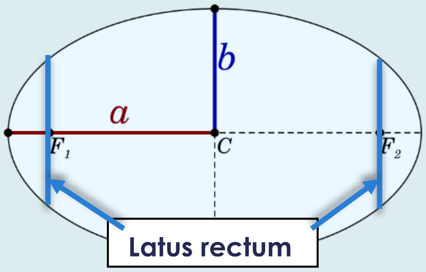
Figure 4"Eccentricidad" by Seahen (CC BY-SA 3.0)

- from the focus (F) to a point on the curve (M)
- perpendicularly from the directrix (M') to that point the ratio between the two distances will stay the same.
- For a circle, the **ratio = 0**
- For an ellipse, **0 < ratio < 1**
- For a parabola, the **ratio = 1** (same distances)
- For a hyperbola, the **ratio > 1**

This ratio is called the **eccentricity**, which means:

"all points whose distance to the focus is equal to the eccentricity times the distance to the directrix[1]"

The **Latus Rectum** is a line that goes through the focus and is parallel to the directrix.



- It is 4 times as long as the focal length in parabolas
- It is the diameter of the circles
- It is $\frac{2b^2}{a}$ in an ellipse where a and b are **half of the major (a) and minor (b) diameter**.

[1] https://www.mathsisfun.com/geometry/conic-sections.html

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

Let's remember some equations for each of these curves:

1. The Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

2. The Circle (where a = b)

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$ and thus, $x^2 + y^2 = a^2$

Notice that the circle is an ellipse with equal diameters.

3. The Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

4. The Parabola

$y^2 = 4ax$ with $a > 0$

We can then find a general formula for all of them since, even if these curves were cut out of a solid, they now belong to plane geometry and their cartesian coordinates can be found.

However, since they are curved and have no straight lines, we cannot only use x and y in this equation.

We will thus need:

- x and y
- x^2 and y^2
- xy
- a constant term
- a factor for each

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

Graphs of quadratic equations in two variables always represent a conic section.

Here is their general equation:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Remember that all the coefficients of this equation need to be real numbers and that A, B and C cannot be all zero.

Let's do some exercises to experiment with these formulas:

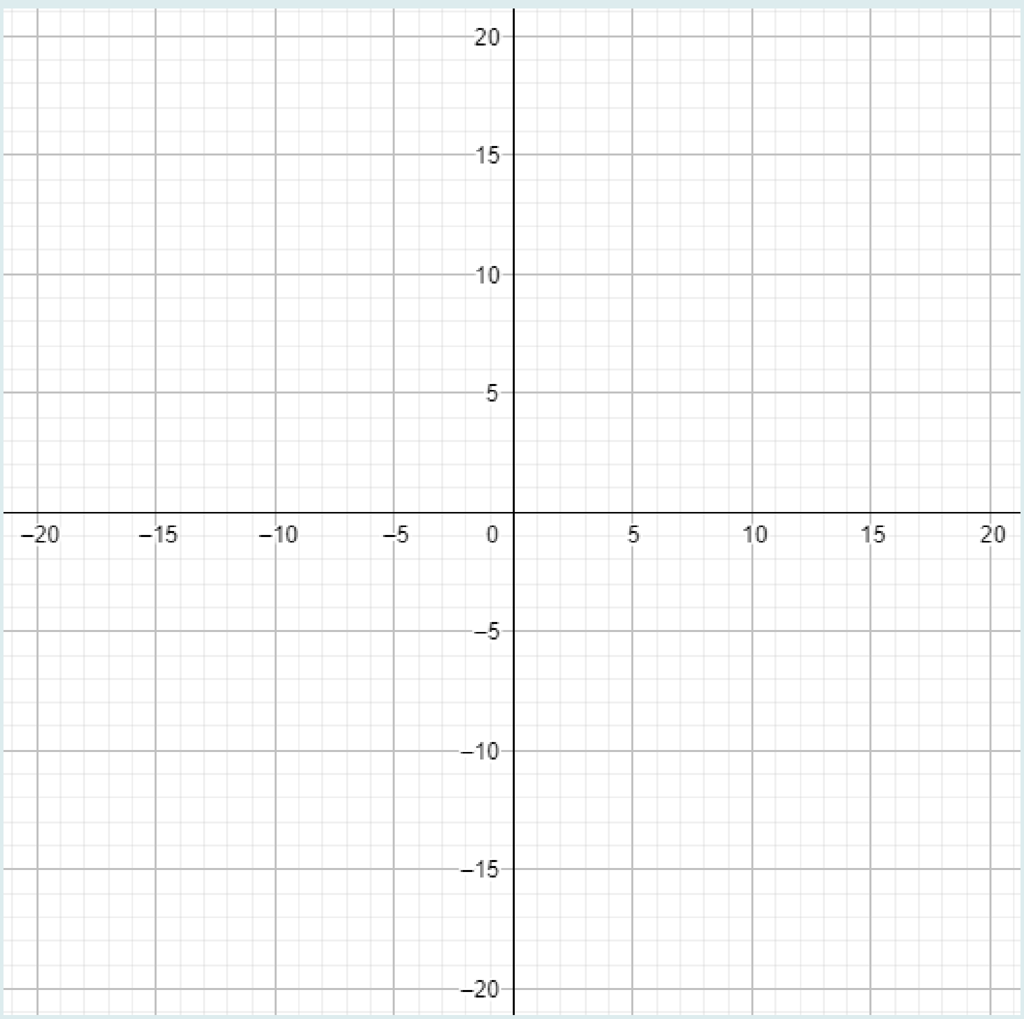
1. Find the equation of the parabola that has its vertex in (0;0) and its focus in (-5;0)

____

____

____

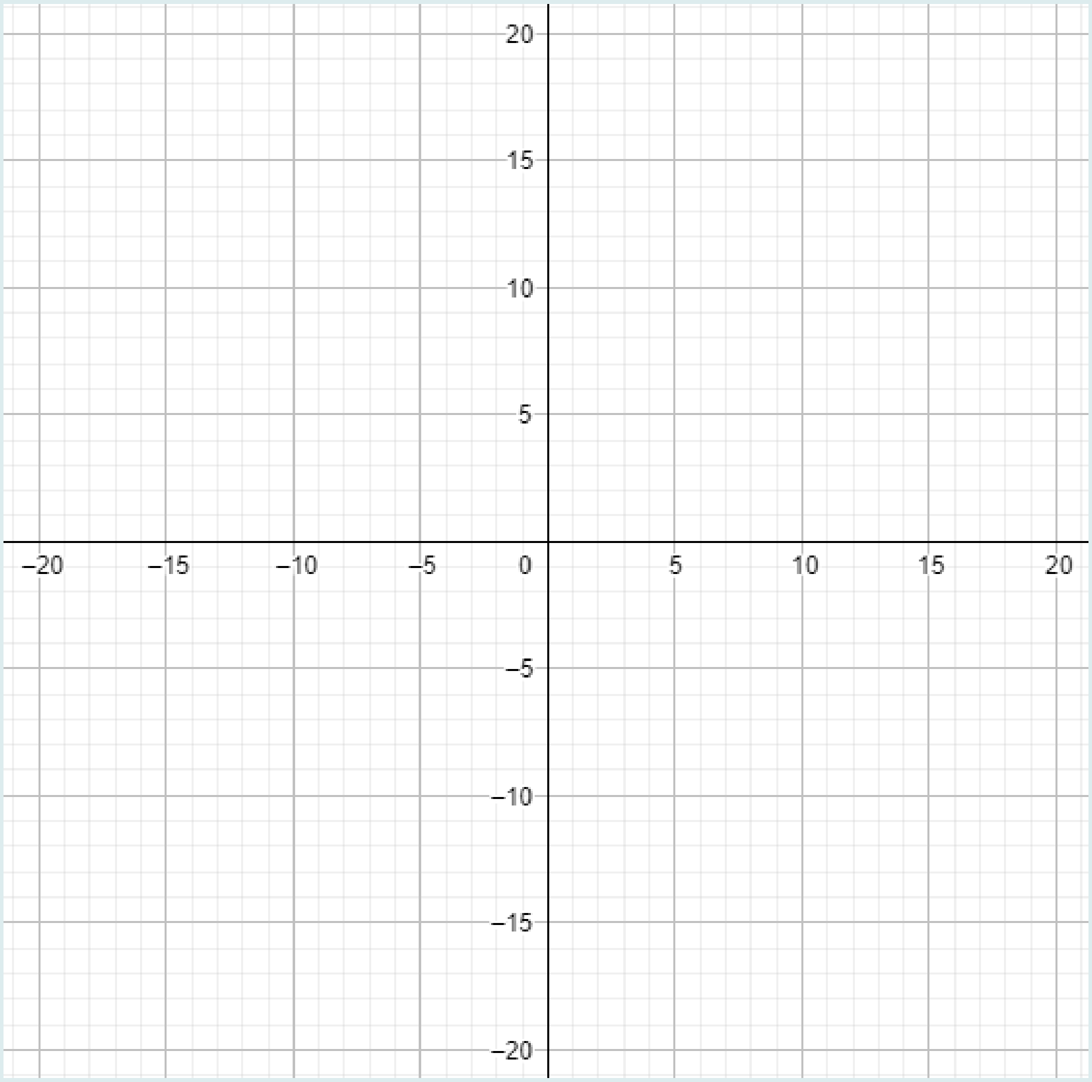
2. Draw it on the following axis:



This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

3. Find the center and the radius of this circle: $x^2 + y^2 - 4x + 8y - 6 = 0$

4. Draw the circle on the following axis:



This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

TASK

Grin like a Cheshire Cat:

In Wonderland, Alice meets a strange cat who slowly vanishes, leaving only its grin. Doesn't it remind you of a shape we just learnt about?


Figure 5: Illustrations from the book "Alice's adventures in Wonderland" by Lewis Caroll

On the first picture, we see its head very clearly, but it seems to have no body. On the second one, we can see it is disappearing in the tree foliage. Its face and grin are still visible, and its chin seems to be resting on one of the tree's branches.

1. Draw the directrix, focus and curve on the picture:

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

2. Answer the following questions:

a) What curve did you find?

____

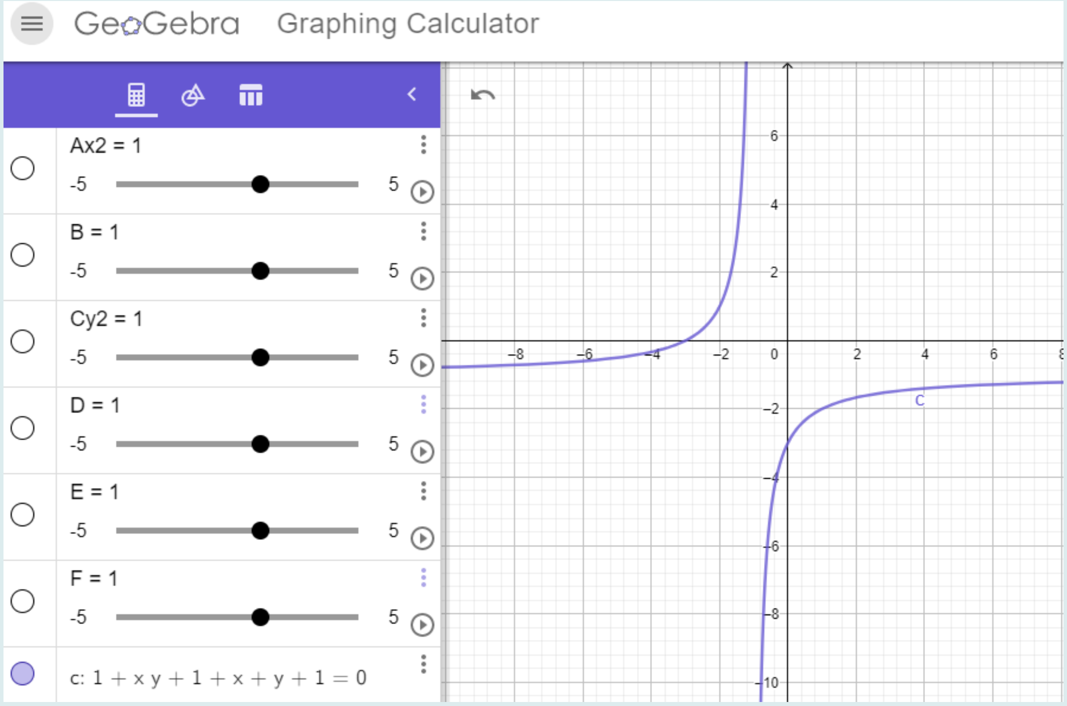
b) Why do you think the author gave the cat this ability to appear and disappear whenever it feels like it?

____

____

3. Use the tool [Geogebra] to draw the shapes more easily:

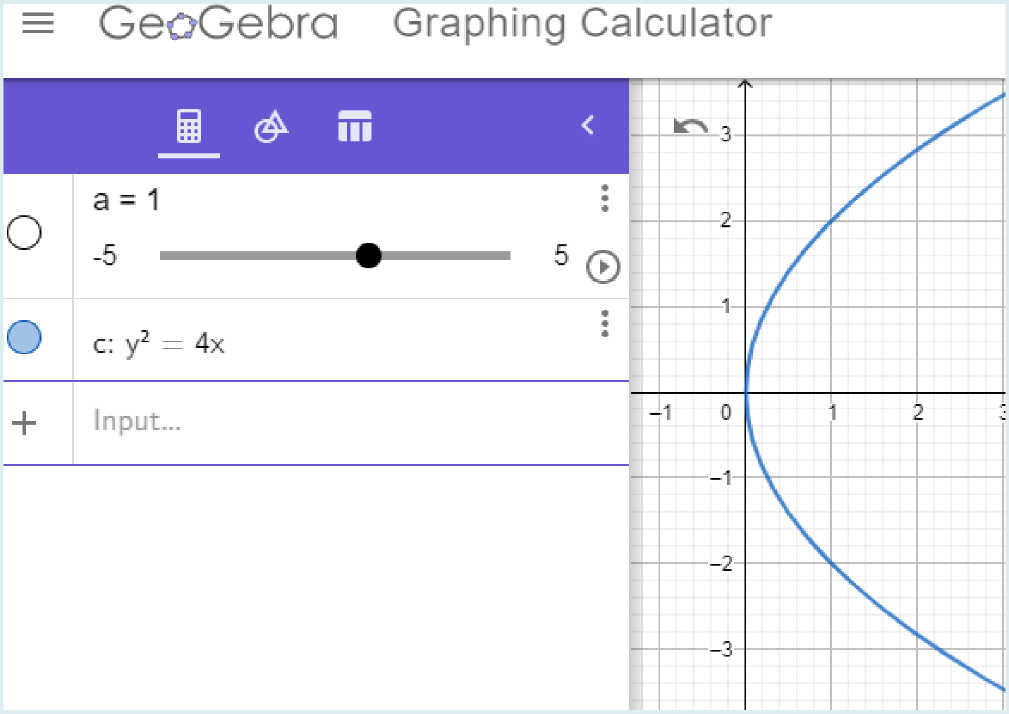
If we type the general equation, we get this:



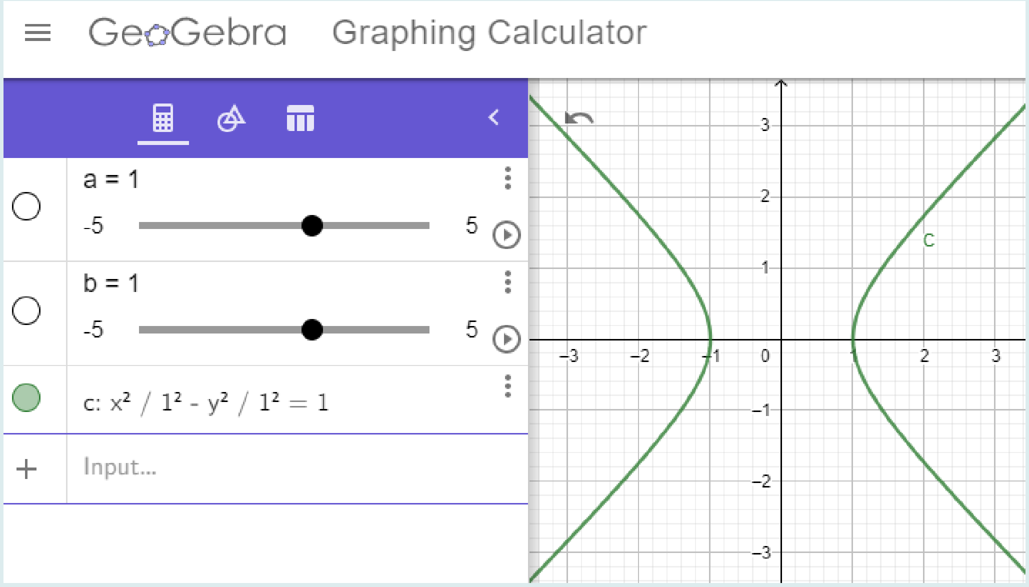
Try to click on the "play" buttons to see what happens 😉

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

Let's write the formula to draw a parabola in the program:



Now, let's try with a hyperbola:

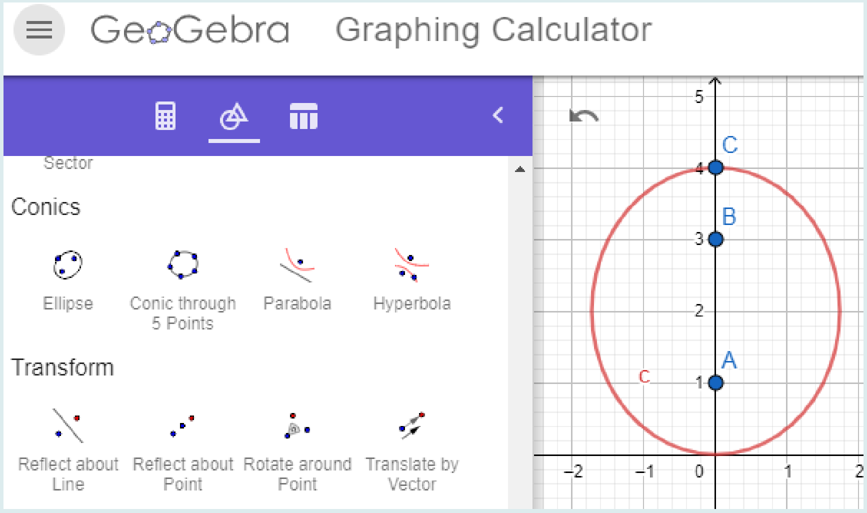


This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

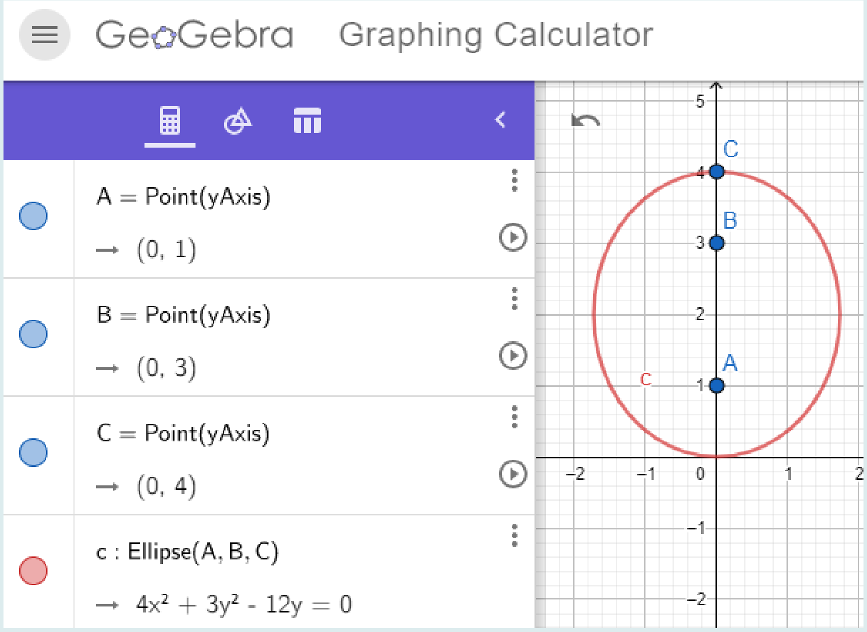
You can also create your own shapes and get their coordinates and equations:

- For the ellipse:

This tool allows you to draw your own ellipse wherever you want on the graph:

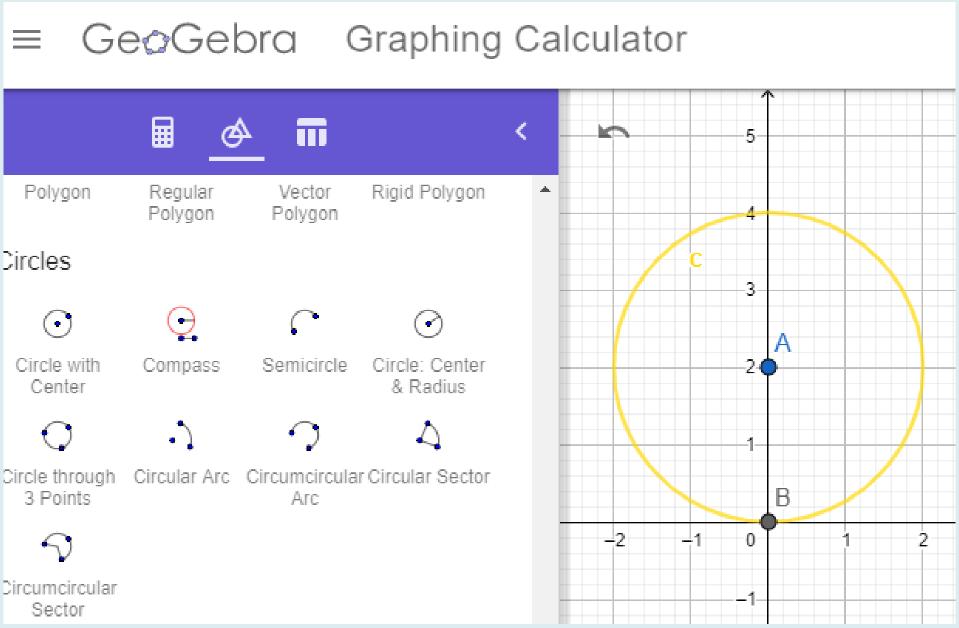


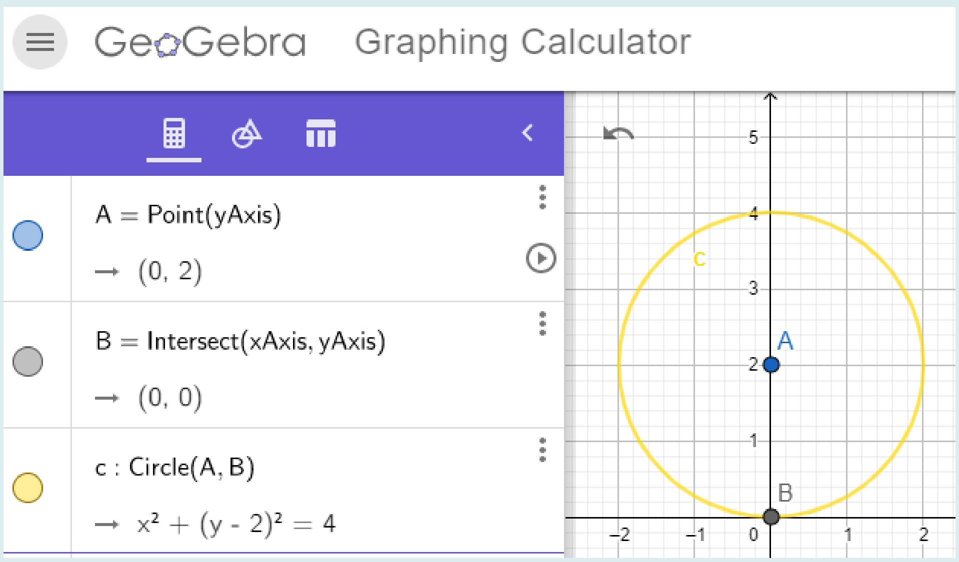
In this part, you can see the coordinates and equations of the produced ellipse:



This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

- For the circle:





When you clicked on the "play" buttons, you saw that the shape could bend, stretch and shrink into other figures.

This is how some mathematicians came to the conclusion that a shape can bend and stretch into other shapes if it keeps its basic properties, just like the Duchess' baby who turned into a pig!

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

LEARN MORE…

The Mathematics of Alice in Wonderland

http://www.massline.org/ScottH/science/MathOfAliceInWonderland-100308.pdf

Algebra in Alice in Wonderland

https://www.newscientist.com/article/mg20427391-600-alices-adventures-in-algebra-wonderland-solved/

Hidden math in Alice in Wonderland:

https://www.maa.org/external_archive/devlin/devlin_03_10.html

Projective Geometry

https://www.britannica.com/science/projective-geometry

Euclid's Postulates in Elements:

https://www.youtube.com/watch?v=LPET_HhN0VM

History of non-Euclidean geometry – part 1:

https://www.youtube.com/watch?v=nkvVR-sKJT8

History of non-Euclidean geometry – part 2:

https://www.youtube.com/watch?v=vUWKMo5scKY

History of non-Euclidean geometry – part 3:

https://www.youtube.com/watch?v=H74AayZkpXg

This project has been funded with support from the European Commission. This publication reflects the views only of the author, and the Commission cannot be held responsible for any use which may be made of the information contained therein.