PART II: Music & Mathematics

AGE RANGE: 16-18

TOOL 22: MUSIC AND FIBONACCI

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TheArtOf Maths

Co-funded by the Erasmus+ Programme of the European Union

Educator’s Guide

Title: Music and Fibonacci

Age Range: 16-18 years old

Duration: 2 hours

Mathematical Concepts: Golden Number, Golden Ratio, Fibonacci’s Sequence

Artistic Concepts: Ancient Greek Music, Muses, Harmony

General Objectives: To discover the mathematical concepts hidden in musical compositions and understand the logical process that lies behind it.

Instructions and Methodologies: The students will explore both fields as a whole, by listening to the music or playing it, and watching the suggested videos that analyze musical composition. They will discover the basis of the mentioned math concepts.

Resources: This tool provides videos and online resources for you to use in your classroom. The topics addressed will be an inspiration for you to personalize and give nuance to your lesson.

Tips for the educator: Learning by doing is very efficient, especially with young pupils with learning difficulties. Encourage them to experiment with a musical instrument, if possible.

Desirable Outcomes and Competences: At the end of this tool, the student will be able to:

- Understand the logical process behind music composition;
- Understand the use of the Golden Ratio and Fibonacci Sequence in music;
- To calculate the n^{th} term of the Fibonacci Sequence.

Debriefing and Evaluation:

Write 3 aspects you liked about this activity:	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

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Introduction

Music and mathematics don't show an obvious connection for those who have never composed or read a music sheet. However, it appears clearly that the timing of musical compositions and the structuring of the sheet by measures evokes a mathematical way of thinking.

Many scholars have studied the implication of mathematics in the arts. Music was one of the focus points of their studies and it was found that, throughout history, many mathematicians had explored that question. Pythagoras, Leonardo Bonacci, and many others have contributed to the research. Different aspects of mathematics, ranging from basic geometry and number sequences to trigonometry, have shown to be used in musical compositions.

Within this tool, we will focus on the applicability of mathematics in musical compositions by first investigating the Golden ratio and Fibonacci's Sequence and exploring the options they offer for music composition.

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The divine proportions in music

In Ancient Greece, people were fond of creative arts, among which music was a very important one since it often accompanied other creative pieces such as theatre or poetry. They believed that each artist was inspired by a muse.

Watch this TED-Ed video which explains this phenomenon with more details: https://www.youtube.com/watch?v=-1aAunaw1GA.

This central role that music played in society was not only inspired by muses. The Greek had also discovered a perfect ratio called "the golden ratio" or "the divine proportion". This ratio is closely linked to Fibonacci's Sequence that you will learn about later. You have probably seen them more than once, even without noticing. You can find them in Ancient architecture as well as in nature and in paintings. What most people don't know is that you can also hear the golden ratio and Fibonacci's Sequence!

Here are two examples of where you can see them:

- The sunflower contains Fibonacci's Sequence;
- This Greek temple has the divine proportions of the golden rectangle.

Figure 1: Sunflower with the Fibonacci Sequence

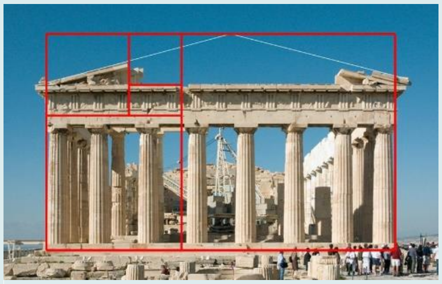

Figure 2: Greek Temple with the Golden Ratio [1]

You can learn more about how Fibonacci's Sequence is represented in nature: https://www.youtube.com/watch?v=_GkxCIW46to.

[1] Image retrieved from: https://www.telitec.com/2019/05/27/golden-ratio/

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Fibonacci's Sequence is based on the perfect proportions of the Golden Ratio, which makes it a great inspiration for musicians to experiment in their compositions. You will find many different interpretations of how the Golden Ratio and the Fibonacci Sequence can be used in musical compositions. Some decide to play a number of notes corresponding to the Fibonacci numbers, some assign those numbers to their piano's tiles,…

You can watch the following videos to hear what music sounds like when linked to the Fibonacci Sequence and the Golden Ratio.

In this video, the musician uses the golden number itself: https://www.youtube.com/watch?v=W_Ob-X6DMI4.

The musicians in this video try to compose based on the Fibonacci Sequence and the Golden Ratio: https://www.youtube.com/watch?v=9mozmHgg9Sk

Glossary

Greek Creative Arts: In Ancient Greek society, the creative arts were history, comedy, poetry, song, hymns, epic poetry, dance, astronomy, tragedy. Each art was believed to be inspired by a Muse.

Muse: In Ancient Greece, Muses were the goddesses that inspired human creativity. Nowadays, we use the word for any person, human or divine, that gives inspiration to an artist in their work.

Harmony: a pleasant musical sound made by different notes being played or sung at the same time[2].

Leonardo Bonacci, or Fibonacci was an Italian mathematician born in the 12th century. He is mostly known for his number sequence, which is found in many aspects of creative arts.

[2] https://dictionary.cambridge.org/dictionary/english/harmony

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The math behind music composition

The Golden Ratio:

The Golden number is a rather unique number in mathematics. It is approximately **1,618** and is often used in art, music, architecture, etc.
We use the Greek letter **φ (phi)** to refer to it.

The golden ratio is the use we make of this number in different disciplines.
Imagine we cut a line in **two** different parts a and b. When we use the golden ratio, the **whole length** divided by the **long side** is equal to the **long side** divided by the **short side**.

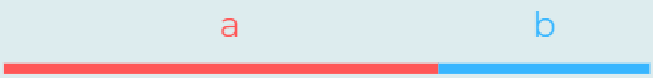

Figure 3: Line divided according to the Golden Ratio

To make it short, remember this formula:

$$\varphi = \frac{(\mathbf{a}+\mathbf{b})}{\mathbf{a}} = \frac{\mathbf{a}}{\mathbf{b}} = \mathbf{1{,}618}$$

The golden ratio can then be applied to a rectangle, called the **Golden rectangle**. As it was seen as the most perfect shape in the Antiquity, many artists and architects used it in their work.

As we have done it with the **line ab**, let's divide a **rectangle AB** into two different parts: a **square A** and a **rectangle B** in which all sides of the square and the long sides of the rectangle have a length of **a** and the short sides of the rectangle a length of **b**.

To have the perfect rectangle, we will use the same formula. Imagine for instance that the square A is 2cm x 2cm. If we want to find the side b:

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We know that:

- $\frac{A}{b} = 1{,}618$
- a = 2

We can say that:

- $\frac{2}{b} = 1{,}618$
- 2 = b x 1,618

If we isolate b, we have:

$$b = \frac{2}{1{,}618} = 1{,}236$$

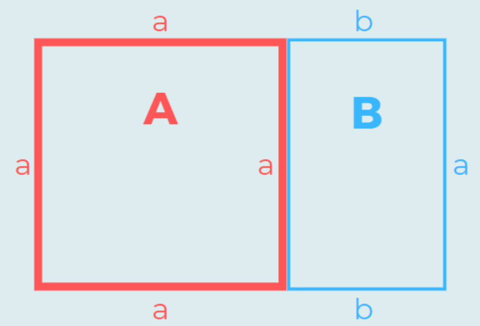
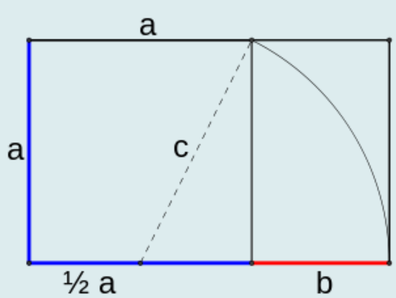


Figure 4: Rectangle divided according to the Golden Ratio

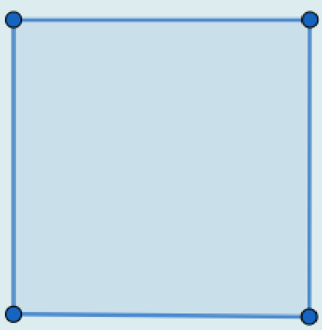
Let's check the result using both formulas:

- $\frac{2 + 1{,}236}{2} = 1.618$
- $\frac{2}{1.236} = 1{,}618$

You can also use a compass and ruler to draw the perfect rectangle:



1. Place your compass' point in the middle of the bottom side
2. Open your compass to touch the opposite angle
3. Draw a curve from the prolongation of the bottom side to its opposite angle.
4. Draw the rectangle B from the start of the curve to the prolongation of the top and bottom sides of square A.



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Fibonacci's Sequence:

The Fibonacci Sequence is a series of numbers where the next number is found by adding the two previous ones.
0+1=1→ 1+1=2 → 1+2=3 → 2+3=5 → 3+5=8 → 5+8=13 ...

The Golden ratio is often associated with the Fibonacci Sequence.

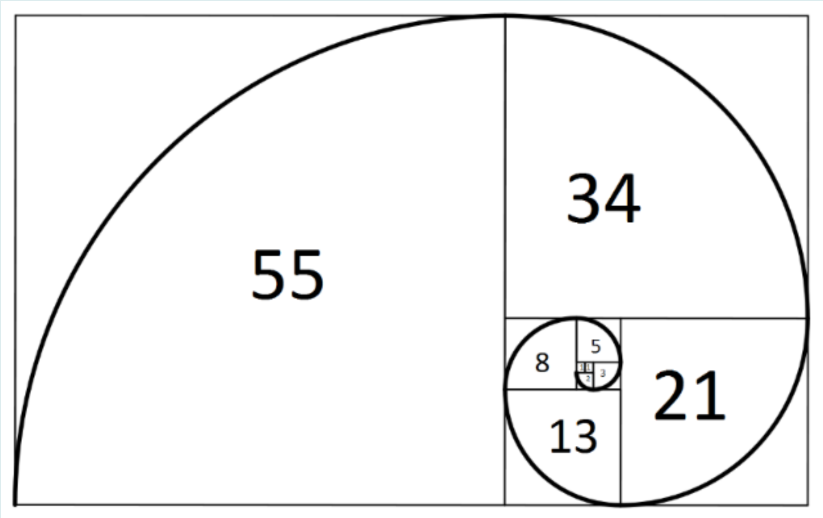
What are the next three numbers?

____

____

____

When we make squares with those widths, we get a nice spiral:



[3]

Figure 5: Representation of the Golden Ratio using Fibonacci's Sequence

Look at how, if we add the sides of squares 5 and 8, we obtain the side of square 13.

Also note that the ratio in the formed rectangles becomes closer and closer to phi.

In the rectangle formed by squares 21 and 13:

a = 21

b = 13

Let's apply the formula: $\frac{a}{b}$ =1,615

[3] https://codegolf.stackexchange.com/questions/53369/fibonacci-spiral

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Let's take the next golden rectangle formed by squares 34 and 21:

a = 34

b = 21

$\frac{a}{b} = 1{,}619$

The results are not exactly the golden number, but they are very close, which shows how Fibonacci's Sequence is connected to the golden ratio!

The sequence can be written in mathematical notation by noticing this:

n =	0	1	2	3	4	5	6	7	8
x_n =	0	1	1	2	3	5	8	13	21

The term number 7 is called $n_7 = 13$

You can remember this rule:

$x_n = x_{n-1} + x_{n-2}$

Where:

x_n is the term number "n"

x_{n-1} is the previous term (n-1)

x_{n-2} is the term before that (n-2)

As the Fibonacci Sequence is very close to the golden ratio, we can use phi to find any number of the sequence with this formula:

$$x_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$

If we look at the numbers of the sequence, we can identify an interesting pattern:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

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We notice that:

- x_3 = 2 and that every third number is a multiple of two (2; 8; 34; 144; 610)
- x_4 = 3 and every fourth number is a multiple of three (3; 21; 144)
- x_5 =5 and every fifth number is a multiple of five (5; 55; 610)

Let's look at the ratios (r) between the numbers:

R1	R2	R3	R4	R5	R6	R7	R8
1	1	1,5	1,666	1,6	1,625	1,615	1,619

We notice that the **odd ratios** (R1, R3, R5, R7) are always **under** the golden number, while the **even ratios** (R2, R4, R6, R8) are always **above** it.

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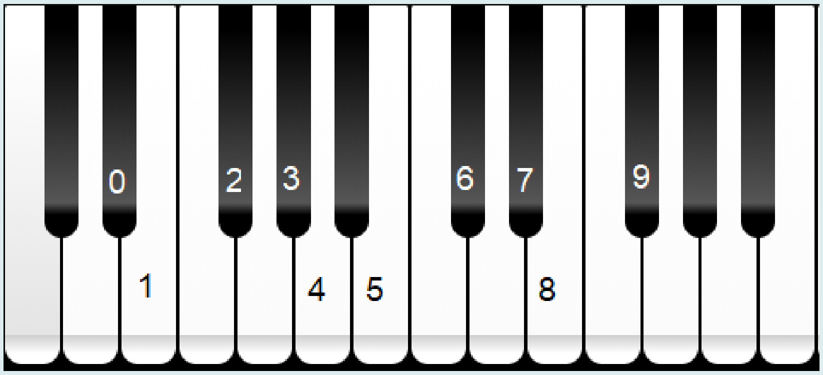
TASK

This task will enable you to use the math you have learnt about the Golden ratio and Fibonacci in a musical composition.

Let's play!

♫ Let's find out the numbers you will have to play on the piano to use the Fibonacci Sequence.

Remember: in this case, the numbers 0 to 9 are located as such on the keyboard.



You will use this online keyboard: https://virtualpiano.net/

or download a virtual piano app on the Play Store or on the App Store

1. Fill in the following table with the numbers you will have to play on the keyboard.

X_{16}	X_{17}	X_{18}	X_{19}	X_{20}

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Use the formula $x_n = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$ to find the missing numbers:

x_{16} =

x_{17} =

x_{18} =

x_{19} =

x_{20} =

You can now play those digits individually on the keyboard to listen to the Fibonacci Sequence!

Result:

This musical experiment by aSongScout shows how to use Fibonacci's Sequence in musical compositions: https://www.youtube.com/watch?v=IGJeGOw8TzQ.

LEARN MORE…

Video about the use of Fibonacci and the Golden ratio in music composition: https://www.youtube.com/watch?v=9mozmHgg9Sk

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Learn how Fibonacci and the Golden ratio are represented in nature:

https://www.youtube.com/watch?v=_GkxCIW46to

How to compose a song with Fibonacci's Sequence and the Golden Ratio:

http://www.faena.com/aleph/articles/how-to-compose-a-song-with-the-golden-ratio-and-the-fibonacci-sequence/

Advice on the use of Fibonacci in musical compositions:

https://www.classicfm.com/discover-music/fibonacci-sequence-in-music/

Paper on the mathematical study of music:

http://eprints.ma.man.ac.uk/1548/1/covered/MIMS_ep2010_103.pdf

Paper on a Fibonacci musical composition:

http://ww.nntdm.net/papers/nntdm-20/NNTDM-20-1-72-77.pdf

TED Talk about how Fibonacci is all around us:

https://www.youtube.com/watch?v=0vVxL60YFJU

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