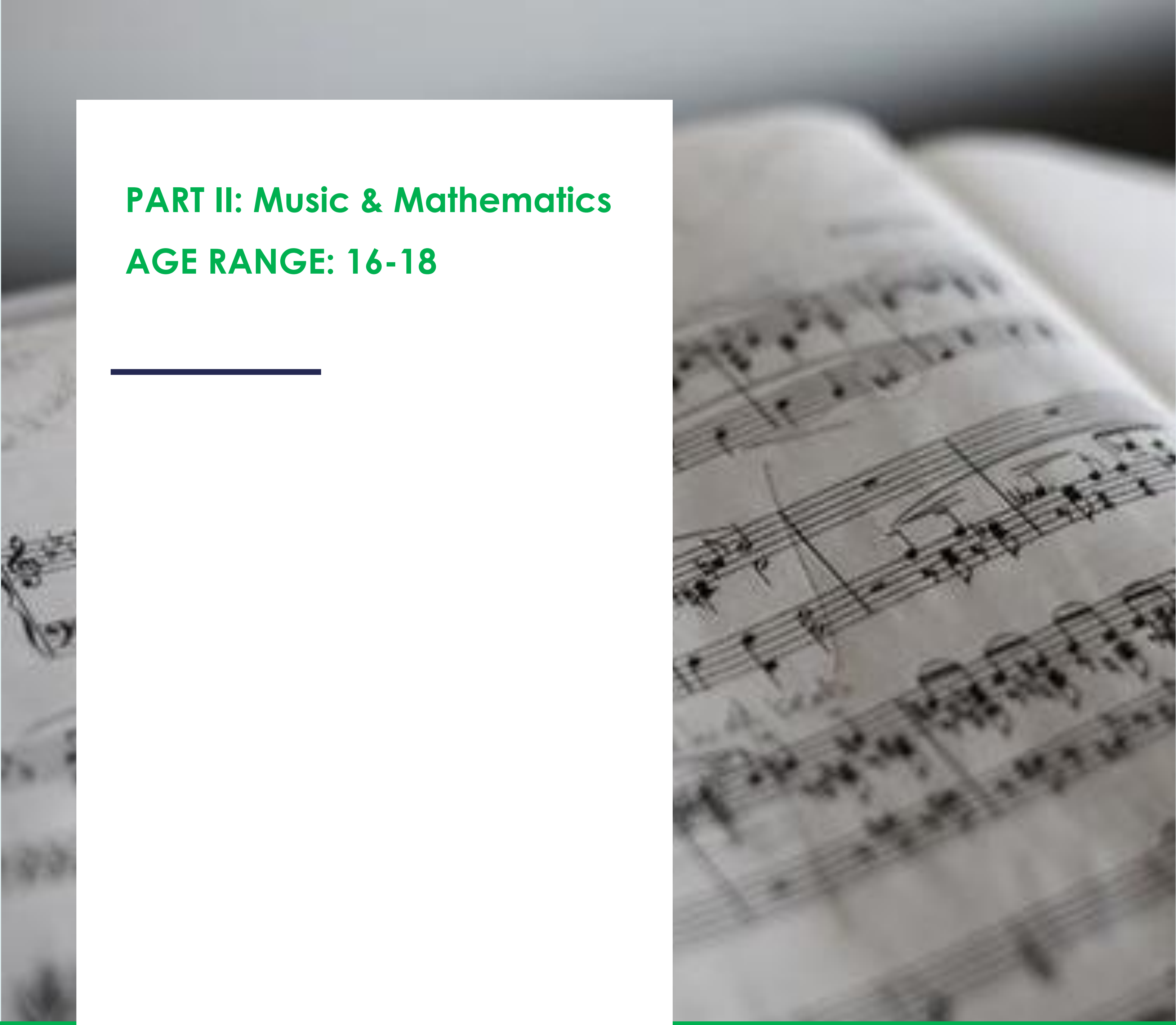

PART II: Music & Mathematics

AGE RANGE: 16-18

TOOL 20: THE BEAT EQUATION

LogoPsyCom

Co-funded by the
Erasmus+ Programme
of the European Union

Educator's Guide

Title: The beat equation

Age Range: 16-18 years old

Duration: 1 hour

Mathematical Concepts: the beat theory, trigonometric identities, sinusoidal waves

Artistic Concepts: Frequency, pitch, sound waves

General Objectives: To discover the mathematical concepts hidden in musical compositions and acquire a more practical view of the use of math.

Instructions and Methodologies: The students will explore both fields as a whole, by listening to the music or playing it and watching the suggested videos that analyze musical compositions. They will discover the basis of the mentioned math concepts.

Resources: This tool provides online resources for you to use in your classroom. The topics addressed in the tool will help you find other materials to personalize and give nuance to your lesson.

Tips for the educator: Learning by doing is very efficient, especially for young learners with learning difficulties. Always explain what each math concept is useful for, practically and create a hands-on experience for them.

Desirable Outcomes and Competences: At the end of this tool, the student will be able to:

- Understand the logical process behind music composition;
- Understand trigonometric identities;
- Understand and use the beat equation.

Debriefing and Evaluation:

Write 3 aspects you liked about this activity:	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

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Introduction

Music and mathematics don't show an obvious connection for those who have never composed or read a music sheet. However, it appears clearly that the timing of musical compositions and the structuring of the sheet by measures evokes a mathematical way of thinking.

Many scholars have studied the implication of mathematics in the arts. Music was one of the focus points of their studies and it was found that, throughout history, many mathematicians had explored that question. Pythagoras, Leonardo Bonacci, and many others have contributed to the research. Different aspects of mathematics, ranging from basic geometry and number sequences to trigonometry, have shown to be used in musical compositions.

Within this tool, we will focus on the applicability of mathematics in musical compositions by first investigating the Pythagorean tuning system and exploring the options it offers for music composition.

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How does music work?

When we play music, the vibration produced, and the movement of air particles goes through our ears and allow us to hear the sounds at the right frequency. If you look at a guitar string, you can see it move in a certain way and at a certain pace. When we stretch a string, its pitch goes higher and its frequency faster. What is produced is called a sound wave and it goes right into our ears, moving the fluid of our cochlea, in the inner part of our ear.

Of course, Pythagoras, a Greek philosopher from around 570 – c. 495 BC was not aware of everything we know today about the human body and musical composition. However, he developed a theory on how to calculate the ratios in intervals, which is what you will learn in this lesson. The legend says he heard different sounds coming from hammers at the blacksmith's shop and found out that when a hammer was twice as big or heavy as another, it produced the same note one octave higher.

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Glossary

Frequency: gives us the speed of a vibration and the pitch of a sound.

Pitch: is whether a note sounds high or low and is measured in Hertz.

Sound wave: represents the vibration produced by a sound. Its length and speed determine the pitch or frequency of the sound.

Cochlea the spiral cavity situated in the inner ear that reacts to sound vibrations.

Interval: is the pitch difference between two sounds.

Octave: is the pitch difference between one note and another that has twice its frequency

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The math behind music composition

The Beat Equation

You learnt that the frequency of a sound is closely linked to the sound waves that we can draw to represent the sounds visually. There is another phenomenon called "the beat", which is produced by the interferences between the two sound waves played at the same time. If you have two sound waves that overlap, you will observe different phenomena.

- If the two sound waves are **constructive**, which means that they are perfectly overlapping, the sound you will hear at that exact moment will be louder.

Figure 1: Constructive sound waves

- If the sound waves are **destructive**, which means that their peaks will be opposed to one another's, the sound you will hear will be **softer**.

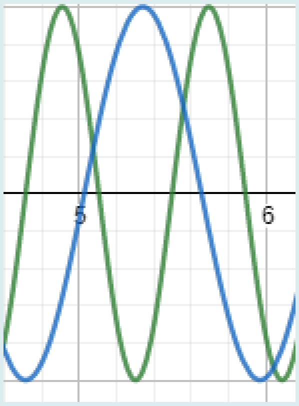

Figure 2: Destructive sound waves

- Both waves being constant, you will hear a regular beat if you listen to them both at the same time.

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Watch this video by SMUPhysics which shows how it sounds like: https://www.youtube.com/watch?v=V8W4Djz6jnY.

Trigonometric Identities

The sound waves can be represented graphically with sinusoidal functions. To do so, you will need to learn about trigonometric identities. We will start with triangles as you already know the Pythagorean theorem.

Here is a right-angled triangle with an angle θ:

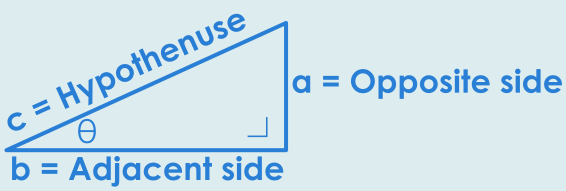

Figure 3: Representation of a triangle to represent trigonometric identities

Here are the trigonometric functions in this triangle:

- $\sin(\theta) = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{a}{c}$
- $\cos(\theta) = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{b}{c}$
- $\tan(\theta) = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{a}{b}$

If we take the Pythagorean Theorem, you will remember this formula to calculate the Hypothenuse: $a^2 + b^2 = c^2$. This formula can be simplified by dividing everything by c^2.

- $\dfrac{a^2}{c^2} + \dfrac{b^2}{c^2} = \dfrac{c^2}{c^2}$
- $\left(\dfrac{a}{c}\right)^2 + \left(\dfrac{b}{c}\right)^2 = 1$

Using the trigonometric functions, we can deduce that:

- $\sin^2(\theta) + \cos^2(\theta) = 1$
- $\cos^2(\theta) = 1 - \sin^2(\theta)$
- $\sin^2(\theta) = 1 - \cos^2(\theta)$

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This is one of the trigonometric identities you need to remember!

Trigonometric identities are complex and there are several formulas that have been demonstrated such as the following one that is used to represent the beat frequency:

$$\mathbf{\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right) * \cos\left(\frac{a-b}{2}\right)}$$

Let's first start with an easy example.

If we decide that a = 5x and b = 4x, here is how it looks if we use a graphing calculator:

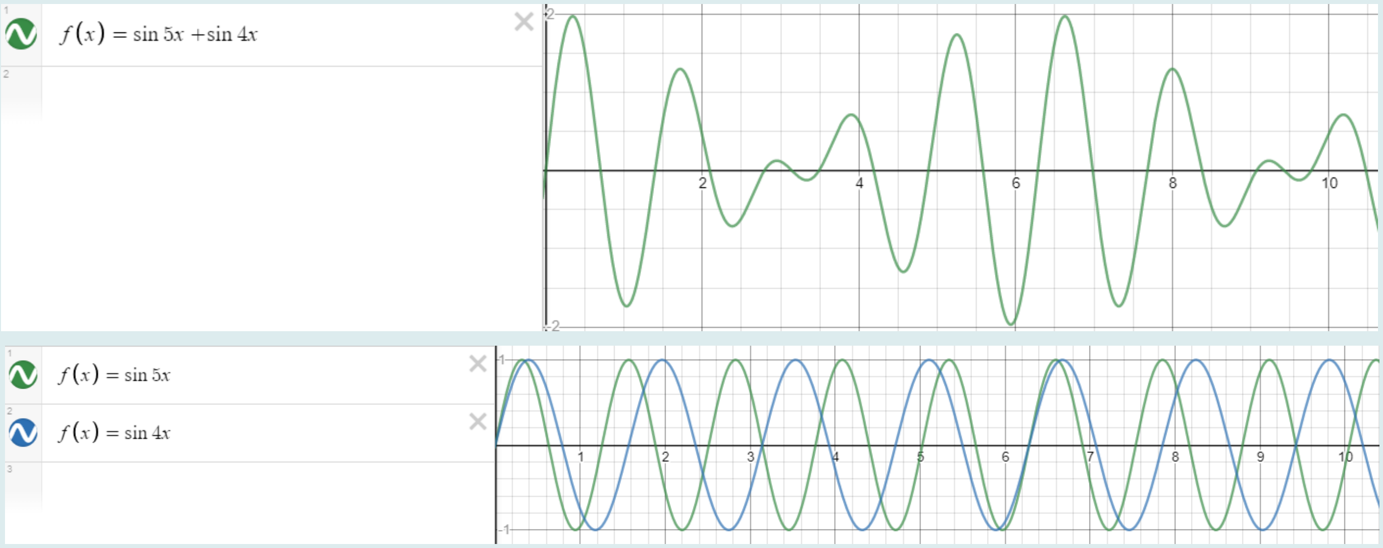

Figure 4: Representation of trigonometric identities

Here is what we see:

- There is a constant cycle
- When two peaks are close to each other, the sound becomes higher
- When a summit and a valley are opposed, the sound gets closer to 0

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Let's see this with a music note. The note A, or La, has a frequency of 440 Hz. Its trigonometric equation is:

$$\sin(440 * 2\pi * x)$$

So, we have:

- a as $(450 * 2\pi * x)$ to have another note close to A/La
- b as $(440 * 2\pi * x)$

Let's apply the beat equation:

- $\sin(450 * 2\pi * x) + \sin(440 * 2\pi * x) = 2\sin(445x * 2\pi) + \cos(5x * 2\pi)$

Here is how it looks like when you put them together using a graphing calculator:

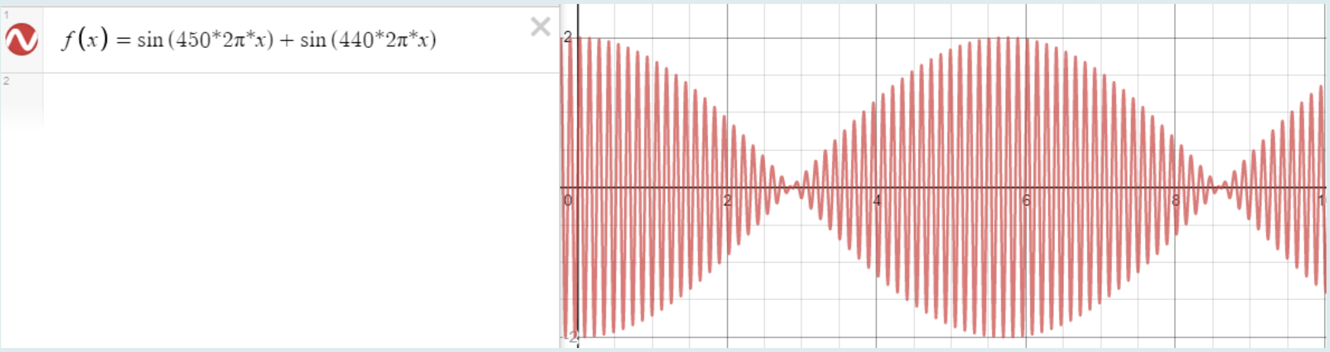

Figure 5: Representation of the beat equation

You can see that, when put together, the waves that appear on the tool show a beat such as the one heard when doing the experiment with two tuning forks.

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TASK

Experience it yourself!

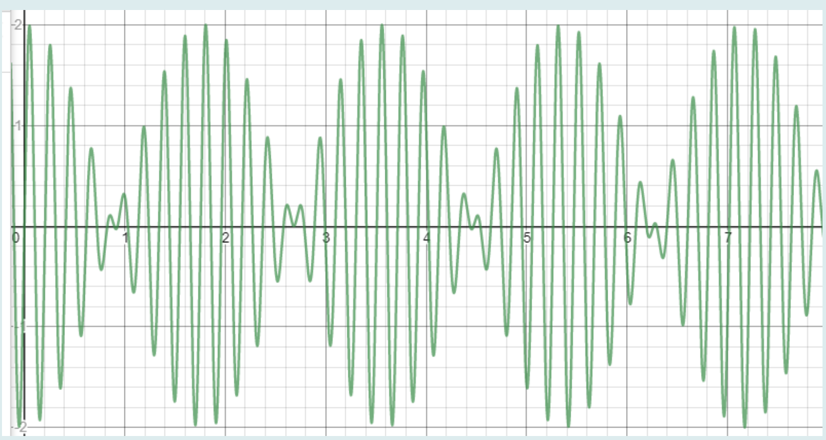
Here is a table with the different frequencies of some music notes:

C	D	E	F	G	A	B
261.63 Hz	293.66 Hz	329.63 Hz	349.23	392 Hz	440 Hz	493.88

Exercise 1: D and C

- What is the beat equation if we play D and C together?

- Take a print screen of the graph on Desmos or GeoGebra:



Exercise 2: A and G

- What is the beat equation if we play A and G together?

- Take a print screen of the graph on Desmos or GeoGebra:

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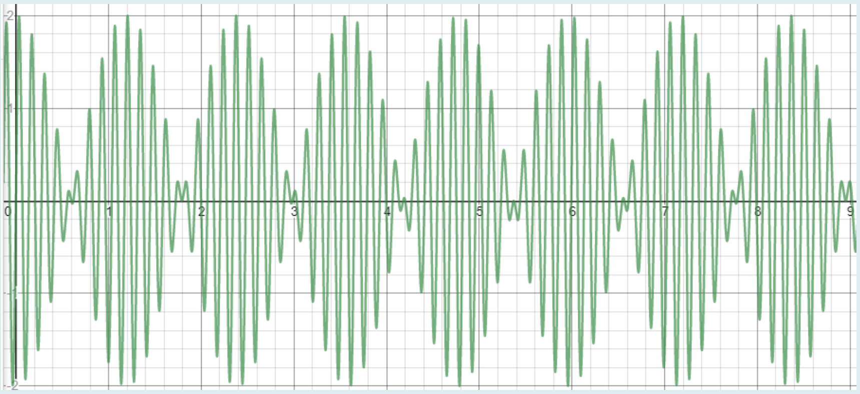

For even more fun, use two tuning forks in your classroom to compare the sounds!

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LEARN MORE…

Ted-ED video about the patterns in music:

https://www.youtube.com/watch?v=zAxT0mRGuoY

Video about where sound comes from:

https://www.youtube.com/watch?v=i_0DXxNeaQ0

Video about how math is used in music:

https://www.youtube.com/watch?v=rTT1XHJKKug

A lesson on the beat frequency:

https://www.youtube.com/watch?v=Ca91iOVGd9A

Video about the physics behind the beat frequency:

https://www.youtube.com/watch?v=IQ1q8XvOW6g

Explanation about the link between trigonometry and music:

http://www-
math.bgsu.edu/~zirbel/sound/Trigonometric%20functions%20and%20sound.pdf

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