

PART I: Visual Arts & Mathematics

AGE RANGE: 16-18

TOOL 11: FRACTALS AND DIMENSIONS

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Educator's Guide

Title: Fractals and dimensions

Age Range: 16-18 years old

Duration: 2 hours

Mathematical Concept: Scaling

Artistic Concepts: Fractals

General Objectives: Fractals are some of the most beautiful and most bizarre geometric shapes. The objectives are to learn and to see some of the mathematics behind these fantastic patterns. The shapes of fractals look the same at various different scales – you can take a small extract of the shape and it looks the same as the entire shape. This curious property is called self-similarity. In the mathematical concept of this tool we will look at polygons inscribed in a circle.

Instructions and methodologies: Read through the text and do the tasks. You will be amazed at the wonders and beauty of the fractals.

Resources: This tool provides you with concepts to teach fractals and the mathematics behind them.

Tips for the educator: Start with the art and move towards the mathematics.

Learning Outcomes and Competences: At the end of this tool, the student will be able to:

- Understand scaling better
- Know the idea of fractals

Debriefing and Evaluation:

Write 3 aspects you liked about this activity:	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

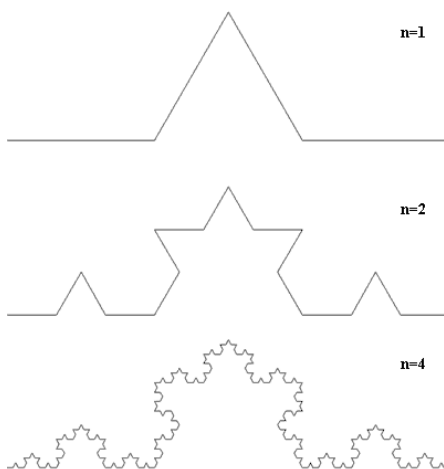
Introduction

Fractals are some of the most beautiful and most bizarre geometric shapes. They look the same at various different scales – you can take a small extract of the shape and it looks the same as the entire shape. This curious property is called self-similarity.

Fractals

To create a fractal, you can start with a simple pattern and repeat it at smaller scales, again and again, forever. In real life, of course, it is impossible to draw fractals with “infinitely small” patterns. However we can draw shapes which look just like fractals. Using mathematics, we can think about the properties a real fractal would have – and these are very surprising.

Fractals are very popular in mathematical visualization, because they look very beautiful even though they can be created using simple patterns like the ones below. You can zoom into a fractal, and the patterns and shapes will continue repeating, forever.



Picture 1 Simple Koch Curves display strict self-similarity:

https://en.wikipedia.org/wiki/File:Fractal_koch.png

Fractal-generating software is any type of graphic software that generates images of fractals. There are many fractal generating programs available, both free and commercial. Mobile apps are available to play or tinker with fractals. Some

programmers create fractal software for themselves because of the novelty and because of the challenge in understanding the related mathematics.

Fractal generating software creates mathematical beauty. Modern computers may take seconds or minutes to complete a single high resolution fractal image. Images are generated for both simulation (modeling) as well as random fractals for art.

Fractal generation used for modelling is used for computer graphics. You can also use it to mimic natural landscapes with fractal landscapes. Fractals are generated in music visualization software.

Glossary

n:

Stands for any natural number.

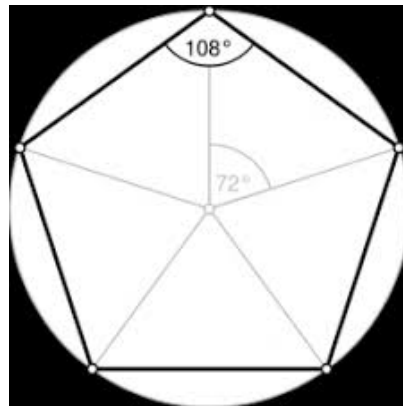
The positive integers the whole numbers examples 0, 1, 2, 3

Polyhedron:

Is a solid in three dimensions with flat polygonal faces, straight edges and sharp corners or vertices.

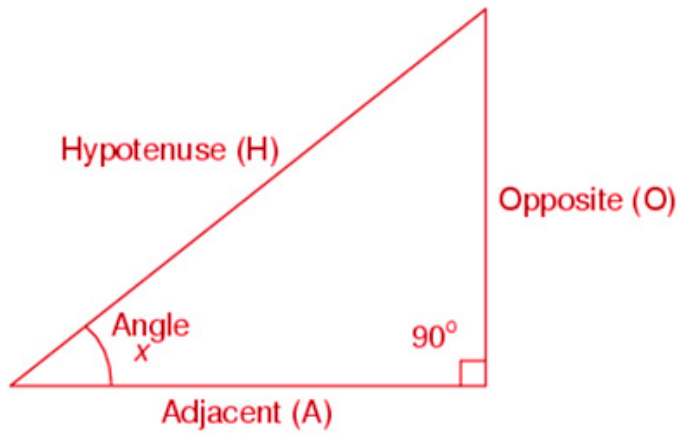
Regular polygon:

A polygon is regular when all angles are equal and all sides are equal (otherwise it is "irregular"). This is a regular pentagon (a 5-sided polygon).

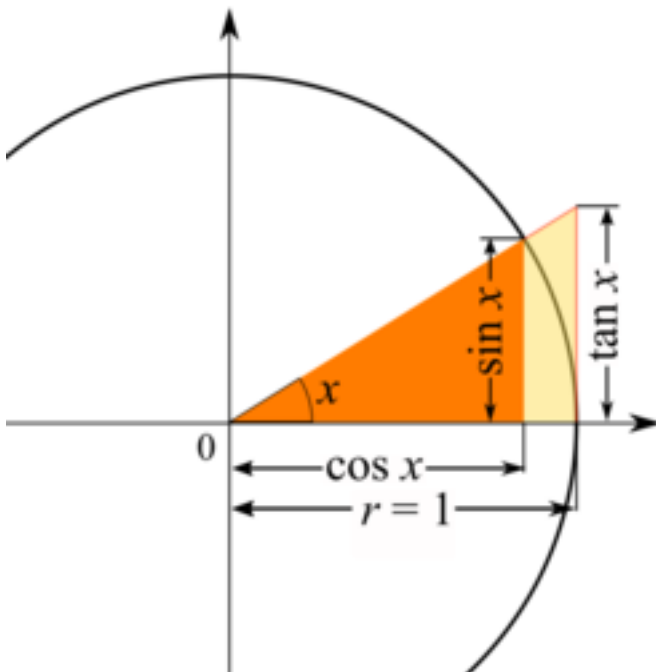


Trigonometry

Sine, Cosine and Tangent are the main functions used in Trigonometry and are based on a Right-Angled Triangle. Before getting stuck into the functions, it helps to give a name to each side of a right triangle as in the picture below:



By the definition above, the trigonometric functions are defined only for angles between 0 and 90 degrees (0 and $\pi / 2$ radians). Using the unit circle, the cosine and sine can be defined as periodic functions with the period 360 degrees (2π radians).



The Math behind fractals



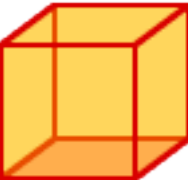

The Sierpinski Gasket

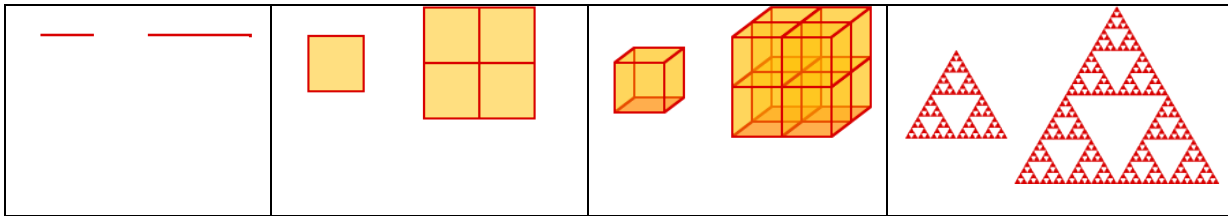
One of the examples is the Sierpinski Gasket consisting of countless triangles. We created it by repeatedly cutting out a triangle in the centre of all the other triangles. However there are many other methods for creating this shape – and here are just a few.



How to do Sierpinski's Gasket in Geogebra

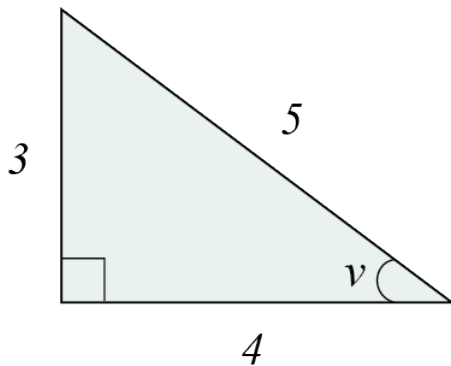
<https://youtu.be/tmmHEa6j-k>

Dimension 1	Dimension 2	Dimension 3	Dimension 1.585...
 ×2	 ×4	 ×8	 ×3
<p>A line has dimension 1. When scaled by a factor of 2, its length increases by a factor of 2.</p>	<p>A square has dimension 2. When scaled by a factor of 2, its area increases by a factor of 4.</p>	<p>A cube has dimension 3. When scaled by a factor of 2, its volume increases by a factor of 8.</p>	<p>When the Sierpinski Gasket is scaled by a factor of 2, its area increases by a factor of 3. Looking at the blue increase factors above, we can deduce it must have a dimension between 1 and 2. There aren't any integers in between 1 and 2, so the dimension of the Sierpinski Gasket has to be fractional. In fact, one can calculate that it is around 1.585.</p>



To Calculate an unknown angle:

Calculate the angle between the hypotenuse and the side that is 4 length units in the following right-angled triangle:



We begin by identifying the angle referred to in the text: it is the acute angle to the right of the triangle. This means that the side that is 4 length units long is the adjacent cathetus and the side that is 3 length units long is the opposing cathetus.

We enter our known values in the formula for cosine:

$$\cos(v) = \text{adjacent cathetus} \div \text{hypotenuse} = 4 \div 5$$

and then we solve the angle v by the inverse function (arccos):

$$v = \cos^{-1}(4/5) \approx 36,87$$

Thus the angle v , whose cosine value is $4/5$, is approximately equal to 36.87 degrees.

We enter our known values in the formula for sine

$$\sin v = \text{opposing catheter} \div \text{hypotenuse} = 3 \div 5$$

and then we solve the angle v by the inverse function (arcsin):

$$v = \sin^{-1}(3/5) \approx 36,87$$

Thus the angle v , whose sine value is $3/5$, is approximately equal to 36.87 degrees.

Finally, we try to do the same thing using the key function:

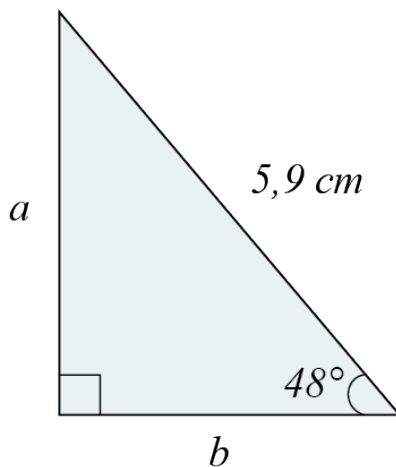
$$\tan v = \textit{opposing catheter} \div \textit{nearby catheter} = 3 \div 4$$

We solve the angle v using the inverse function (arctan) and get thus the angle v , whose tangent value is $3/4$, is approximately equal to 36.87 degrees. As we can see, we get the same value at the angle v regardless of which of the two trigonometric functions we choose to use, which is completely in order.

Calculate the unknown length of a page:

Say that we have a right-angled triangle with a known angle of 48° , a hypotenuse with a length of 5.9 cm and that we want to calculate the lengths of the catheti.

To begin with, we should draw a figure so that we get an overview of the sides and angles of the triangle, thereby reducing the risk of us having to reason wrongly:



From the known angle, side b is the adjacent cathetus. Since we know the length of the hypotenuse, we use the cosine function to determine the length of side b :

$$\begin{aligned} \cos(48^\circ) &= b \div 5,9 \\ 5,9 * \cos(48^\circ) &= \\ b &\approx 3,948 \end{aligned}$$

From our figure we can see that side a is the opposite side, so we use the sine function to find the length of side a (in this situation we could have also used the key function, since we now know the length of the nearby cathetus):

$$\begin{aligned}\sin 48^\circ &= a \div 5,9 \\ 5,9 * \sin 48^\circ &= a \div 5,9 * 5,9 \\ 5,9 * \sin 48^\circ &= a \\ a &\approx 4,385\end{aligned}$$

TASKS

The Greek mathematician and physicist Archimedes of Syracuse (287-212 B.C) used in and-described polygons to determine an approximate value of π .

TASK 1

Show that when calculating the area of a regular polygon with n sides inscribed on a circumference with any radius' length, an approximate value of $\pi \approx n * \sin (180^\circ/n)$ is obtained, whereas when calculating the area of a regular polygon with the same number of sides circumscribed in a circle with the same radius, an approximate value of $\pi \approx n * \tan (180^\circ / n)$ is obtained.

Note you just showed that $n*\tan(180^\circ/n) > \pi > n*\sin(180^\circ/n)$.

TASK 2

Choose $n = 96$ and compare with the values that Archimedes got on the geometric path.

TASK 3

Examine by choosing larger and larger values of n how well you can close the value of π using your calculator.

LEARN MORE...



How to draw fractals under 5 minutes:

<https://youtu.be/sFEYQMrWNHU>