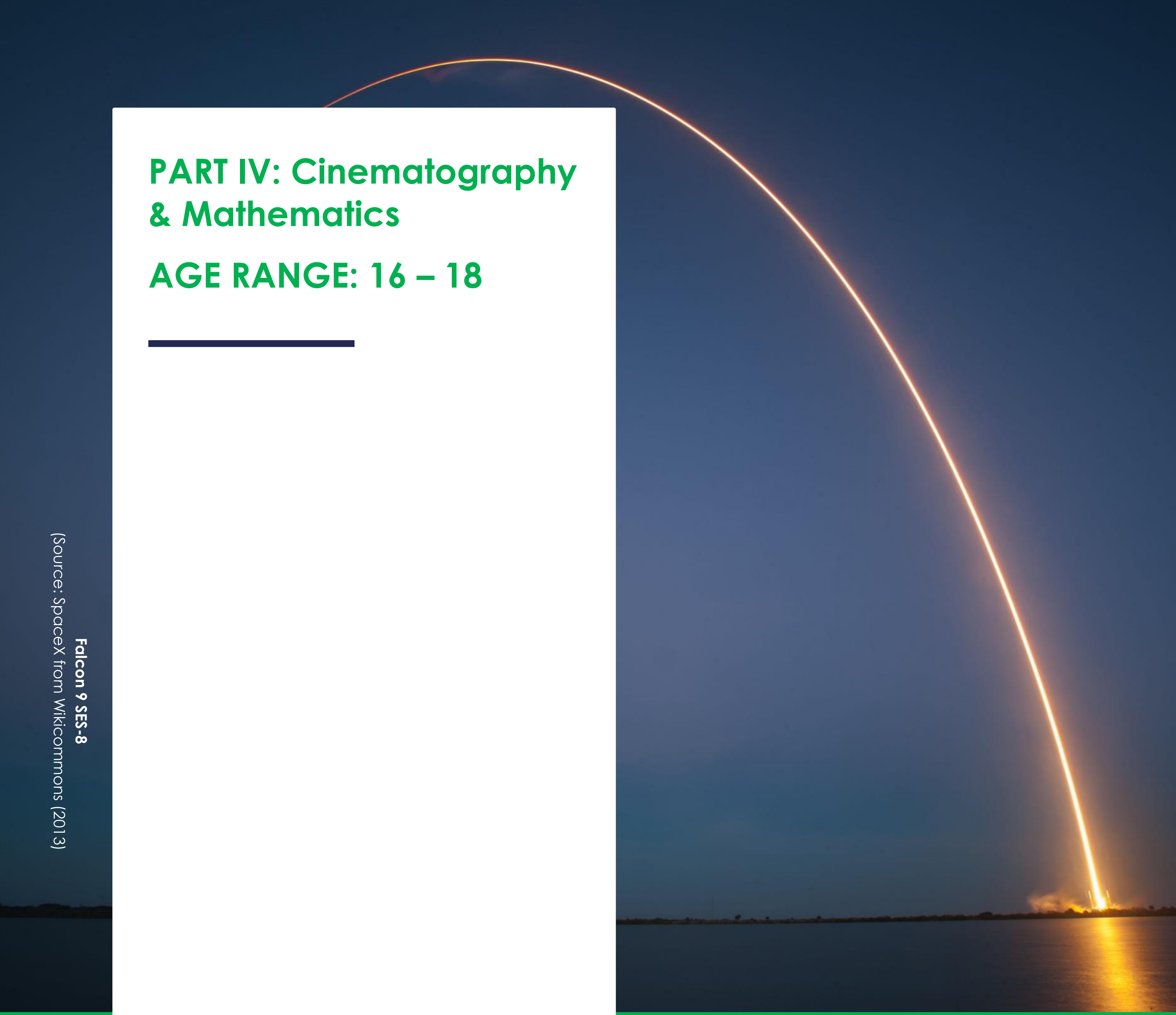
PART IV: Cinematography & Mathematics

AGE RANGE: 16 – 18



Falcon 9 SES-8
(Source: SpaceX from Wikicommons (2013)

TOOL 43: QUADRATIC FUNCTION THROUGH THE MOVIE "OCTOBER SKY"

SPEL – Sociedade Promotora de Estabelecimentos de Ensino

TheArtOf Maths

Co-funded by the Erasmus+ Programme of the European Union

Educator's Guide

Title: Quadratic Function through the movie "October Sky"

Age range: 16 – 18 years old

Duration: 2 hours

Mathematical concepts: Quadratic function

Artistic concepts: Free falling, Vertical Launch

General objectives: To understand the notions of the quadratic functions, calculate the coordinates of its vertex, solve quadratic equations and inequalities. Calculate the position of a projectile when free falling or thrown vertically.

Instructions and Methodologies: Show the excerpt of the movie October Sky in which the quadratic function is referenced (cf. link on "Searching the NET") and suggest students to watch the full movie at home; Use a graphic calculator (for instance, the online graphic calculator Desmos) to show students graphs, as well as the results of quadratic equations / inequalities.

Resources: Computer with an internet connection; Access to the website: https://www.desmos.com/

Tips for the educator: Begin by getting the graphs of some quadratic functions to explain their properties. Give an example for each of the concepts taught and then let the students practice similar exercises.

Learning Outcomes and Competences: At the end of this tool, the student will be able to:

- Predict a quadratic's function appearance and obtain its results.
- Calculate the maximum or minimum point of a quadratic function;
- Solve quadratic equations and inequalities;
- Calculate the position of a projectile when free falling or thrown vertically.

Debriefing and Evaluation:

Write 3 aspects you liked about this activity:	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

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Introduction

Sometimes we find aspects related to Mathematics in television series or movies. In such cases, sometimes these Mathematical concepts are not given much importance, because they do not influence the story itself. However, there are a few cases in which they do.

Some examples include: "21" (USA, 2008), by Robert Luketic; "Proof" (USA, 2005), by John Madden; "A Beautiful Mind" (USA, 2001), by Ron Howard; "Enigma" (USA, 2001), by Michael Apted; "Pi" (USA, 1988), by Darren Aronofsky; "Good Will Hunting" (USA, 1997), by Gus Van Sant and "Cube" (Canada, 1997), by Vincenzo Natali.

In this tool, the movie "October Sky" (USA, 1999,) by Joe Johnston, will be discussed and its mathematical concepts, such as projectile trajectories and the quadratic function, will be covered.

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October Sky

October Sky (1999) is an American biographical drama film based on the Homer Hickman's novel "Rocket Boys"[1]. It is based on a true story of Homer Hickam, a son of a coal miner, who, against his father's will, was inspired to launch homemade rockets when Sputnik 1 was first launched in October, in 1957, by the Soviet Union.

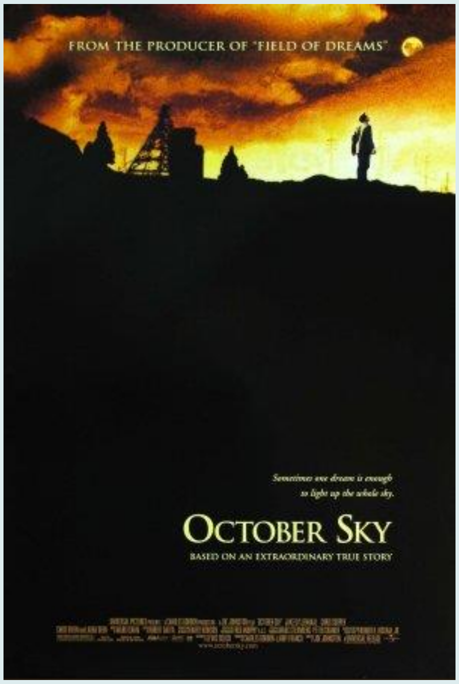

Fig. 1 – October Sky (1999) movie poster
(Source:https://pt.wikipedia.org/wiki/October_Sky)

From free fall to vertical launching of rockets, in October Sky, Homer and his friend Quentin use such concepts in order to prove their innocence of a fire that started close to the location in which one of the rockets that they had launched crashed. To do so, they rely on the quadratic function to demonstrate it was impossible for a projectile to fall in such location.

Eventually, Homer Hickman was hired as an aerospace engineer by NASA.

[1] In order to reach to a wider public, "October Sky", an anagram of "Rocket Boys" was used instead.

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Glossary

Free fall: any motion of a body in which gravity is the only responsible for causing its acceleration.

Sputnik 1: Earth's first artificial satellite launched by the Soviet Union on October 4, 1957, in the Soviet Union rocket test unit currently known as Baikonur Cosmodrome.

Projectile: any object that is moving under the influence of gravity.

Vertical launch: the act of launching a body upwards or downwards, which, unlike free fall, has an initial velocity value.

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Maths behind October Sky

Just like Homer and his friend calculated in what area the rocket they had launched crashed, it is possible to calculate the height of an object that is free falling (dropped) or thrown, as a function of time. To do so, the following algebraic function is used:

$$\mathbf{h(t) = \pm\frac{1}{2}gt^2 + v_0t + h_0}$$

Where:

$\mathbf{h(t)}$ -> height (in meters)

$\mathbf{t}$ -> time (in seconds)

$\mathbf{v_0}$ -> initial velocity (in m/s)

$\mathbf{h_0}$ -> initial height (in meters)

g -> gravitational acelaration in m/s^2 (whose approximate value on Earth is **9,8**)

Therefore:

$$\mathbf{h(t) = \pm\frac{1}{2}gt^2 + v_0t + h_0} \Leftrightarrow$$

$$\mathbf{h(t) = \pm\frac{1}{2}(9,8)t^2 + v_0t + h_0} \Leftrightarrow$$

$$\mathbf{h(t) = \pm 4,9t^2 + v_0t + h_0}$$

Note: Due to the fact that when an object is launched top-bottom its acceleration is positive, whereas when it is launched bottom-up its acceleration is negative, when an object is in free fall or is vertical launched from top to bottom (that is, dropped), the plus sign (+) is used in the formula; when an object is vertically launched from the bottom up, the minus sign (−) is used.

Note 2: If the height is given in feet, the formula takes the following form:

$$\mathbf{h(t) = \pm\frac{1}{2}(32)t^2 + v_0t + h_0 \Leftrightarrow h(t) \pm 16t^2 + v_0t + h_0}$$

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These are called are **quadratic functions** and the graphs that result from it are called **parabolas**.

Let us take a closer look at the quadratic functions in order to learn some of their properties.

Quadratic function

Definition and characteristic elements

A **quadratic function**, or polynomial function of the second degree, is a function f defined by:

$$\mathbf{f(x) = ax^2 + bx + c}, \qquad a \neq 0$$

Where:
$\mathbf{a}, \mathbf{b}$ and c are **real numbers.**

The domain of a quadratic function is the set of real numbers.

The graph of a quadratic function is a curve called a **parabola**.

Examples:

$y = 2x^2$ $\qquad$ $y = -x^2 + 4$ $\qquad$ $y = x^2 + 6x + 4$

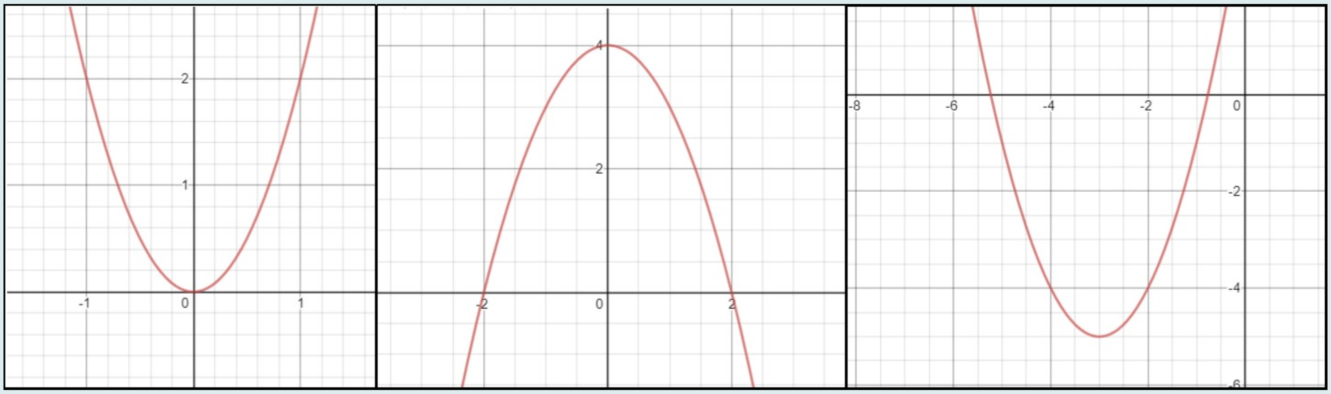

Fig. 2 - Quadratic Functions
(Source: Author; https://www.desmos.com/)

- The value of the parameter a influences the curve of the graph:
  - if $a > 0$, the parabola is U-shaped, **with an opening at the top**;
  - if $a < 0$, the parabola is U-shaped, **with an opening at the bottom**.

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A parabola has particular characteristics:

- A parabola is symmetrical with respect to a vertical line, which is called the axis of symmetry.
- The axis of symmetry of a parabola is the vertical line of equation $\boldsymbol{x = -\frac{b}{2a}}$.
- The point of intersection of the parabola with the axis of symmetry is called a vertex.

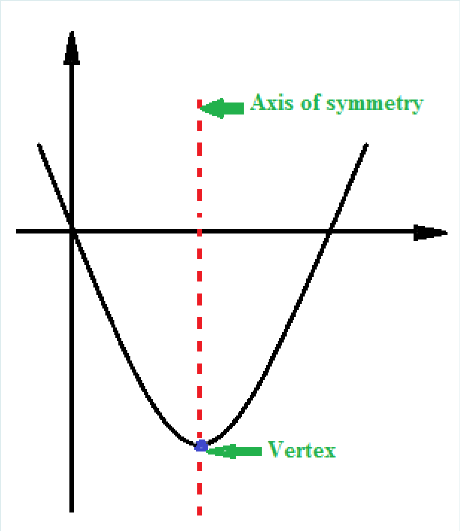

Fig. 3 - Axis of symmetry
(Source:http://calculator.mathcaptain.com/vertex-calculator.html)

- The coordinates of the vertex of a parabola are given by the expressions $\mathbf{x_V = -\frac{b}{2a}}$ and $\mathbf{y_V = f(x_V) = f\left(-\frac{b}{2a}\right)}$ or $\mathbf{y_V = \frac{4ac-b^2}{4a}}$.

Roots of a quadratic function

Roots or **zeros** of the quadratic function, $\mathbf{f(x) = ax^2 + bx + c;\ a, b, c} \in \mathbb{R}$ R and a $\neq$ 0 to the real numbers $\boldsymbol{x}$ such that $\boldsymbol{f(x) = 0}$.

Therefore, the roots of the function $\mathbf{f(x) = ax^2 + bx + c,\ a \neq 0}$ are the solutions of the second-degree polynomial equation $\mathbf{ax^2 + bx + c = 0}$, which are given by the so-called **quadratic formula:**

$$\boldsymbol{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

We have:

$$\boldsymbol{f(x) = 0 \Leftrightarrow ax^2 + bx + c = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

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Examples:

Consider the zeros of the function defined by $\boldsymbol{f(x) = x^2 - 6x + 5}$

- Identify each of the values: a = 1, b = - 6 e c = 5
- Use the quadratic formula:

$$\boldsymbol{f(x) = 0 \Leftrightarrow x^2 - 6x + 5 = 0 \Leftrightarrow x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4 \times 1 \times 5}}{2 \times 1}}$$

$$\boldsymbol{\Leftrightarrow x = \frac{6 \pm \sqrt{16}}{2} \Leftrightarrow x = \frac{6 \pm 4}{2} \Leftrightarrow}$$

$$\boldsymbol{x = \frac{6-4}{2} \vee x = \frac{6+4}{2} \Leftrightarrow x = \frac{2}{2} \vee x = \frac{10}{2} \Leftrightarrow x = 1 \vee x = 5}$$

Roots = {1, 5}

Note:

1) The number of real roots of a quadratic function depends on the value obtained for the n^th root Δ= $b^2 - 4ac$, called discriminant, namely:

   - when Δ> 0, there are **two real** (and distinct) **roots**;
   - when Δ = 0, there is **one real root** (to be more precise, there are two equal roots);
   - when Δ <0, there are **no real roots**.

2) Some polynomial equations of the second degree can be solved without using the quadratic formula. This happens, for instance, when second-degree equations are incomplete, that is, of the type:

   - $\boldsymbol{ax^2 + c = 0}$;
   - $\boldsymbol{ax^2 + bx = 0}$.

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Intersection of the graph of a quadratic function with the coordinate axes

Intersection of the graph with the axis Oy

To obtain the coordinates of the intersection point of the graph of the function $\boldsymbol{f}$ defined by por $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{a}\boldsymbol{x}^2 + \boldsymbol{b}\boldsymbol{x} + \boldsymbol{c};\ \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}$ and $\boldsymbol{a} \neq \boldsymbol{0}$ with the axis $\boldsymbol{Oy}$, replace x by 0. Since $\boldsymbol{f}(\boldsymbol{0}) = \boldsymbol{c}$, there is always an intersection point of the graph of the quadratic function with the y-axis. The coordinates of the intersection point are $(\boldsymbol{0}, \boldsymbol{c})$.

Intersection of the graph with the axis Ox (zero of a function)

A quadratic function can have one zero, two zeros, or none. Let us take a look the following graphical representations of quadratic functions.

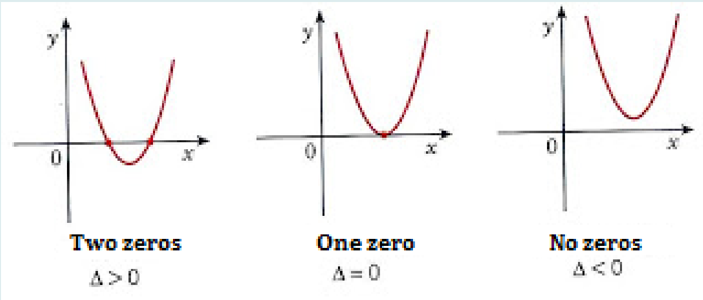

Fig. 4 – Graphical representation of quadratic functions
(Source: http://funcaode2grau.blogspot.com/2008/02/resumo-terico-da-funo-quadrtica.html)

The abscissa of the zero ordinate points are the zeros of the function.

When it comes to the quadratic function, identifying its zeros is of interest both for the resolution of second-degree equations and inequalities, as well as for the resolution of real context problems.

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Finding signs (+/-) of the roots of a quadratic function

- **If the function has two real roots (Δ> 0)**

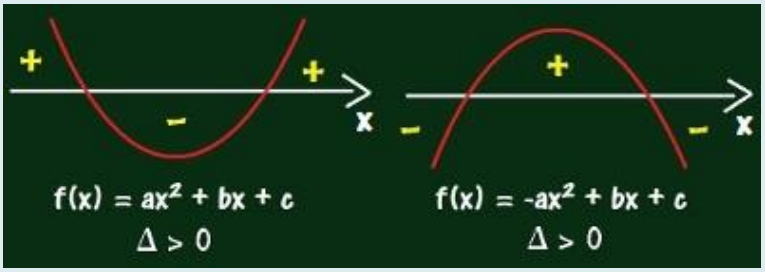

Fig. 5 – Sign of a quadratic function with 2 zeros
(Source: https://alunosonline.uol.com.br/matematica/estudo-variacao-sinal-uma-funcao-2-grau.html)

Positive: $]-\infty; \boldsymbol{x_1}[\cup]\boldsymbol{x_2}; +\infty[$

Negative: $]\boldsymbol{x_1}; \boldsymbol{x_2}[$

Positive: $]\boldsymbol{x_1}; \boldsymbol{x_2}[$

Negative: $]-\infty; \boldsymbol{x_1}[\cup]\boldsymbol{x_2}; +\infty[$

- **If the function has one real root (double equal roots)** $(\Delta = \mathbf{0})$

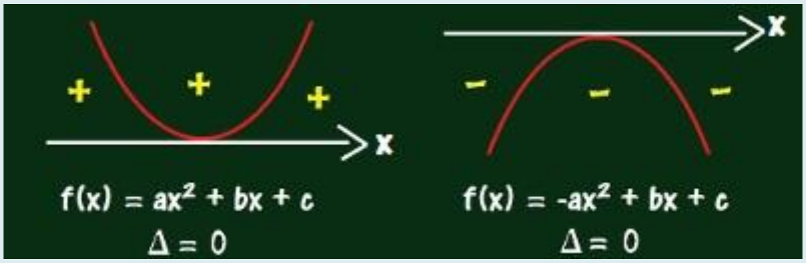

Fig. 6 – Sign of a quadratic function with 1 zero
(Source: https://alunosonline.uol.com.br/matematica/estudo-variacao-sinal-uma-funcao-2-grau.html)

Positive: $\mathbb{R}\backslash\{\boldsymbol{x_1}\}$ $\boldsymbol{or}$ $]-\infty; \boldsymbol{x_1}[\cup]\boldsymbol{x_1}; +\infty[$

Negative: $\{\quad\}$ $\boldsymbol{or}$ $\emptyset$

Positive: $\{\quad\}$ $\boldsymbol{or}$ $\emptyset$

Negative: $\mathbb{R}\backslash\{\boldsymbol{x_1}\}$ $\boldsymbol{or}$ $]-\infty; \boldsymbol{x_1}[\cup]\boldsymbol{x_1}; +\infty[$

- **If the function has no real roots** $(\Delta < \mathbf{0})$

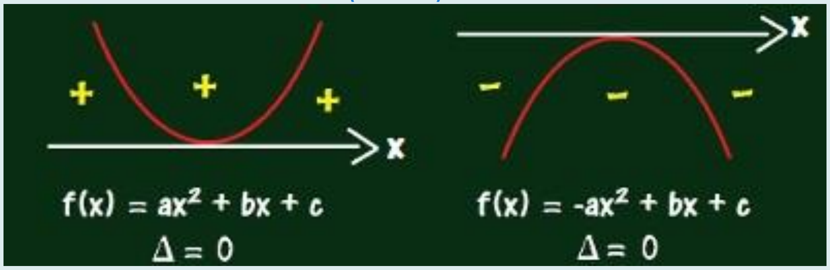

Fig. 7 – Sign of a quadratic function no zeros
(Source: https://alunosonline.uol.com.br/matematica/estudo-variacao-sinal-uma-funcao-2-grau.html)

Positive: $\mathbb{R}$

Negative: $\{\quad\}$ $\boldsymbol{or}$ $\emptyset$

Positive: $\{\quad\}$ $\boldsymbol{or}$ $\emptyset$

Negative: $\mathbb{R}$

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Second-degree Inequalities

In the canonical form of a second-degree equation $ax^2+bx+c=0$, where $a\neq 0$, if the "equal sign" (=) is replaced by a "not equal to" sign (≠) it becomes a **second-degree inequality.**

Solving a second-degree inequality consists of determining the values of x that correspond to the positive, negative, non-positive or non-negative values of the function $y=ax^2+bx+c$.

Example: Solve the inequality $x^2+2x>3$

Resolution:

- 1st step: $x^2+2x>3 \Leftrightarrow x^2+2x-3>0$

- 2nd step: Solve the equation $x^2+2x-3=0$

$$x^2+2x-3=0 \Leftrightarrow x=\frac{-2\pm\sqrt{2^2-4\times1\times(-3)}}{2\times1} \Leftrightarrow x=\frac{-2\pm\sqrt{16}}{2}$$

$$\Leftrightarrow x=\frac{-2\pm4}{2} \Leftrightarrow x=\frac{-2-4}{2} \vee x=\frac{-2+4}{2} \Leftrightarrow x=\frac{-6}{2} \vee x=\frac{2}{2} \Leftrightarrow \quad x=-3 \vee x=1$$

- 3rd step: Make the graphical representation of $f(x)=x^2+2x-3$ and identify the zeros and the areas where the function is positive and negative.

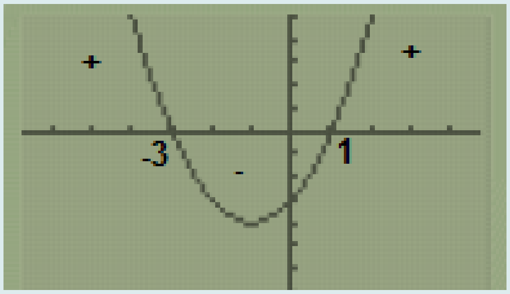

Fig. 8 – Graphic of the function $f(x)=x^2+2x-3$
(Source: Graphing Calculator)

- 4th step: Write the solution set the inequality.

$$x^2+2x>3 \Leftrightarrow x^2+2x-3>0 \Leftrightarrow x\in\,]-\infty;-3[\,\cup\,]1;+\infty[$$

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Note: The previous inequality can be also represented using the online graphic calculator Desmos.

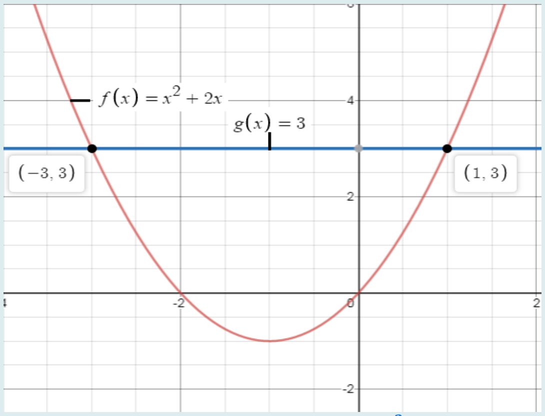

Fig. 9 – Graphic of the inequality $x^2 + 2x > 3$
(Source: https://www.desmos.com/calculator)

In this case, the solution set corresponds to the part of the graph of the quadratic function that is above the line of equation $\boldsymbol{x = 3}$, ou seja, $\boldsymbol{C.S.} =]-\infty; -3[\cup]1; +\infty[$

The quadratic function represented as $y = a(x - h)^2 + k$

For any quadratic function to be represented as $\boldsymbol{y = a(x - h)^2 + k}$, where $\boldsymbol{(h, k)}$ are the coordinates of its vertex, you simply need to know the coordinates of its vertex and one more of its points.

Quadratic functions in projectile launching

As mentioned previously, the following algebric function allows to calculate the height of an object when free falling or launched vertically, at a given time:

$$\boldsymbol{h(t) = \pm 4,9t^2 + v_0 t + h_0}$$

Where:

$\boldsymbol{h(t)}$ -> height (in meters)

$\boldsymbol{t}$ -> time (in seconds)

$\boldsymbol{v_0}$ -> initial velocity (in $\boldsymbol{m/s}$)

$\boldsymbol{h_0}$ -> initial height (in meters)

$\boldsymbol{4,9}$ -> gravitational acelaration on Earth in $\boldsymbol{m/s^2}$ (represented by the letter **g**)

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Remember that when an object is in free fall or is vertical launched from top to bottom the plus sign (+) is used in the formula, whereas if the object is vertically launched from the bottom up, we use the minus sign (−).

Let's take a look at some examples:

To facilitate the calculation, consider the universal gravitational constant, $\boldsymbol{g}$, equals $\mathbf{10}\ \boldsymbol{m/s^2}$, which results in the following formula:

$$\boldsymbol{h(t) = \pm 5t^2 + v_0 t + h_0}$$

Examples:

1) A projectile is dropped 80m from the ground. If $g = \mathbf{10}\ \boldsymbol{m/s^2}$ and it is free of any dissipative forces, determine the moment of when the object will hit ground.

Resolution:

➔ 1st step: Write the formula that corresponds to how high the projectile was when it was dropped.

The object was dropped at 80m high, hence its initial height is $\boldsymbol{h_0} = \mathbf{80}\ \boldsymbol{m}$. Since the initial velocity is $\boldsymbol{v_0} = \mathbf{0}\ \boldsymbol{m/s}$ and the projectile is going from top-bottom, the formula will be:

$$\mathbf{h(t) = 5t^2 + 0 \times t + 80 \ \Leftrightarrow h(t) = 5t^2 + 80}$$

➔ 2nd step: Solve the equation $\mathbf{h(t) = 0}$

$$\mathbf{h(t) = 0 \Leftrightarrow 5t^2 + 80 = 0 \Leftrightarrow 5t^2 = 80 \Leftrightarrow t^2 = \frac{80}{5} \Leftrightarrow t^2}$$
$$\mathbf{= 16 \Leftrightarrow \quad t = \pm\sqrt{16} \Leftrightarrow t = \pm 4 \Leftrightarrow t = 4\ s.}$$

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➔ 3rd step: Answer: The projectile hits the ground after 4 seconds.

2) A projectile is launched vertically from the ground, with a velocity of 72 km/h. Find out:

a) the function that gives us the height of the projectile;

c) the maximum height reached;

d) the height for t = 3 s, and the direction of movement at that time;

e) the moment when the object reaches the ground.

Obs.: Consider $g = 10 \text{ m/s}^2$

Resolution:

a) Converting 72 km/h in m/s results in 20 m/s. Since the initial velocity is $v_0 = 0\, m/s$, that the initial height is $h_0 = 0\, m$ and that the projectile was launched from bottom-up, the formula is $\mathrm{h(t)} = -5\mathrm{t}^2 + 20 \times \mathrm{t} + 0 = -5\mathrm{t}^2 + 20\mathrm{t}$.

b) The maximum height reached is the vertex of the parabola. The following formula is used to calculate the coordinates of the vertex of the parabola: $\mathrm{t} = -\frac{\mathrm{b}}{2\mathrm{a}} = -\frac{20}{2\times(-5)} = 2$. This means that highest point is reached after 2s. Therefore, the highest point is $\mathrm{h}(2) = -5 \times 2^2 + 20 \times 2 = 20$ m.

c) $h(3) = -5 \times 3^2 + 20 \times 3 = 15$ m. Up until 2 s the movement is directed upwards (maximum height) and for $t > 2s$ the movement is directed downwards, meaning that at 3 seconds the object is descending.

d) The time of descent is equal to the time of ascent, therefore the object will hit the ground after 2 + 2 = 4 s. This result could be reached by solving the equation $\mathrm{h(t)} = 0$.

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TASKS

TASK 1

A bullet is placed 1.5 m above the ground and is thrown at a certain angle to ground level. The trajectory of the bullet is given by the function defined by:

$$\mathbf{f(x) = -0,0025x^2 + x + 1,5}$$

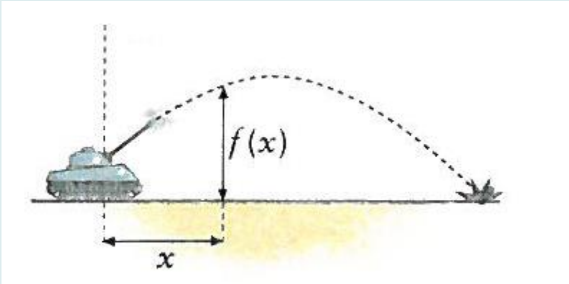

Fig. 10 - Trajectory of a projectile
(Source: Neves, M. A, Pereira, A., Leite, A., Guerreiro, L., & Silva, M. C. (2006). Matemática A2 – Ensino Profissional: Funções polinomiais. Porto: Porto Editora.)

Where $f(x)$ is the height of the bullet (in meters) and x the horizontal distance from the bullet to the point of launch.

1.1 Determine the horizontal distance in meters, to one decimal place, between the launch point and the point where the bullet fell.

1.2. Determine the maximum height reached by the bullet and how far it landed.

TASK 2

Solve, analytically, the following inequality $2x^2 - 8x > -6$.

TASK 3

A ball is thrown vertically from the bottom up.
The height h, in meters, where the ball is t seconds after the launch is defined by:

$$\mathbf{h(t) = 1 + 38t - 5t^2.}$$

3.1 Determine $h(0)$ e and interpret the result in the context of the presented situation.

3.2 Determine the maximum height reached by the ball and the moment it occurred.

3.3 At what time did the ball hit the ground? Present the answer to one decimal place.

3.4 At what interval of time was the ball less than 30 meters away from the ground? Present the answer to the decimal place of a second.

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LEARN MORE...

October Sky (1999) movie plot

https://www.imdb.com/title/tt0132477/?ref_=nv_sr_1?ref_=nv_sr_1

Quadratic functions on October Sky

https://www.youtube.com/watch?v=udHB3tftPz4

Forms & features of quadratic functions

https://www.khanacademy.org/math/algebra/quadratics/features-of-quadratic-functions/v/rewriting-a-quadratic-function-to-find-roots-and-vertex

Sign of a Quadratic Function with Application to Inequalities

http://www.sosmath.com/algebra/quadraticeq/signquadra/signquadra.html

Quadratic Word Problems: Projectile Motion

https://www.purplemath.com/modules/quadprob.htm

Explore graphs of trigonometric functions with Desmos web application

https://www.desmos.com/

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