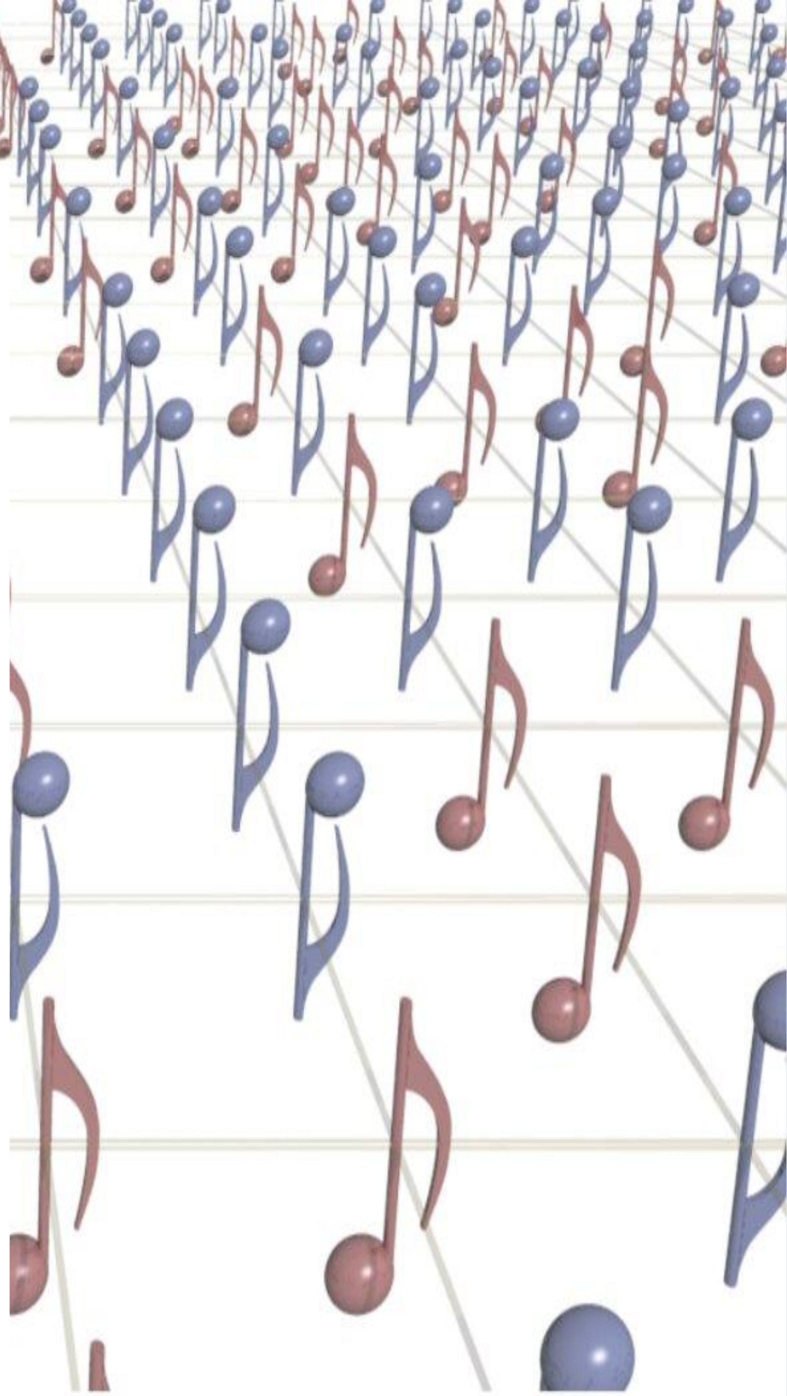
PART II: Music & Mathematics

AGE RANGE: 16-18



Transições da música
(Source: https://phys.org/news/2019-05-phase-transitions-math-

TOOL 27: LOGARITHMS IN THE TEMPERED SCALE

SPEL – Sociedade Promotora de Estabelecimentos de Ensino

Co-funded by the Erasmus+ Programme of the European Union

Educator's Guide

Title: Logarithms in the Tempered Scale

Age range: 16-18 years old

Duration: 3 hours

Mathematical concepts: Logarithms, Properties of logarithms, Rules of Operation of Logarithms

Artistic concepts: 12-tone equal temperament, musical notes and frequencies of musical notes.

General objectives: To understand the logarithm concept, the properties of a logarithmic function and perform operations involving logarithms.

Instructions and Methodologies: It will be useful to use a scientific calculator (it can be the online graphing calculator Desmos) so the student can learn how to calculate exponents in a calculator and verify the tasks' results/solutions.

Resources: Computer with an internet connection; Access to the website: https://www.desmos.com/

Tips for the educator: Begin by giving examples on how to calculate logarithms, with increasing difficulty, while explaining its properties and changes according to their base.

Learning Outcomes and Competences:

At the end of this tool, the student will be able to:

- Obtain the graphic of a logarithmic function;
- Calculate the value of logarithms with different bases.

Debriefing and Evaluation:

Write 3 aspects you liked about this activity:	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

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Introduction

Mathematics and Music have always been connected. However, the first evidence of this relationship was only found in the sixth century BC that the first evidence of this relationship were discovered. Pythagoras compared the sound produced by hammers of different lengths, used by blacksmiths, to the sound of the monochord, of which it is believed Pythagoras was the inventor.

This comparison allowed Pythagoras to discover and improve the mathematical reasons behind the sounds through the study of the sounds produced by the monochord. He divided the string into two equal parts, then into three equal parts, and so on. He matched the sounds mathematically according to the subdivisions he was making and created the Pythagorean scale, in which each note maintained a well-defined relationship with the other.

The Pythagorean scale is the basis of the diatonic scale, consisting of seven notes, which is the basis of the formation of all the other scales used in Western music. One of the scales that emerged in Western culture was the 12-tone equal temperament, known as tempered scale or chromatic scale, in which there is a greater consonance between notes.

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Erasmus+

Exponents in the Tempered Scale

In the sixth century BC, Pythagoras used the monochord to study the relationship between the length of the vibrating string and the musical tone it produces. Imagine a string stretched and fixed both ends. When an end of this string is touched, it vibrates and produces a note that is called a fundamental note. Pythagoras then divided the string into two equal parts, then into three and so on. As he continued to subdivide the string, obtaining the harmonics of the fundamental note, and, by mathematically combining the sounds, he created scales which resulted in notes that were naturally related to each other.

By keeping the same intervals (numerical ratio of $\frac{3}{2}$) between the notes and by starting from the octave interval given by the frequencies $\mathbf{f_0}$and $\mathbf{2f_0}$, the Pythagorean diatonic scale can be formed. The notes obtained, commonly known as C, R, E, F, G, A and B, otherwise represented in most countries by the solfège naming convention Do-Re–Mi–Fa–Sol–La–Ti (or Si) in accordance to the correspondence C-Do, D-Re, E-Mi, F-Fa, G-Sol, A-La and B-Ti (or Si), form the so-called diatonic scale of seven notes that for centuries was the basis of other scales.

From the Middle Ages onwards, it was noticeable that some notes were too close to each other (for example, the notes B and C) so a decision has been made to create a scale in which the frequency interval between all notes would be the same ratio. The value of it being the interval between the notes C and the B (a semitone). As a result, the 12-tone equal temperament was formed and improved by J. S. Bach.

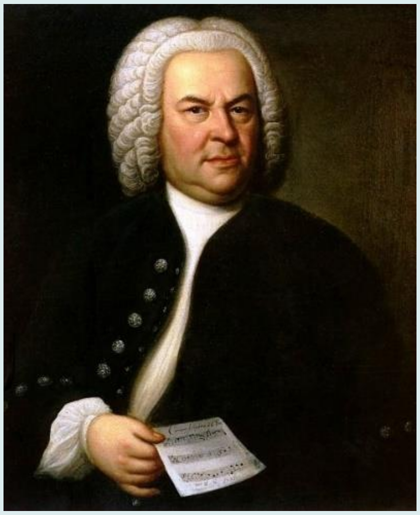

Fig. 1 - Johann Sebastian Bach
(Source:https://commons.wikimedia.org/wiki/Johann_Sebastian_Bach)

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Unlike Pythagoras who had formed the diatonic scale by obtaining 7 notes through a division that can be represented by fractions, this new temperament can be explained by the use of logarithms, a concept introduced by John Napier (1550-1617), and resulted in 12 notes: C, C#, D, D#, E, F, F#, G, G#, A, A# and B.

Note	Semitone	Pythagorean scale	Equal Tempered scale	Consonance
C	0	1	1	1
C#	1		$2^{1/12}$	
D	2	$9/8$	$2^{2/12}$	
D#	3		$2^{3/12}$	$6/5$
E	4	$81/64$	$2^{4/12}$	$5/4$
F	5	$4/3$	$2^{5/12}$	$4/3$
F#	6		$2^{6/12}$	
G	7	$3/2$	$2^{7/12}$	$3/2$
G#	8		$2^{8/12}$	
A	9	$27/16$	$2^{9/12}$	$5/3$
A#	10		$2^{10/12}$	
B	11	$243/128$	$2^{11/12}$	
C (octave)	12	2	2	2

Table 1: Octave division in equal temperament

In other words, the 12 notes of the tempered or chromatic scale correspond to the logarithms of base 2: $\mathbf{2^0}$, $\mathbf{2^{\frac{1}{12}}}$, $\mathbf{2^{\frac{2}{12}}}$, …, $\mathbf{2^{\frac{2}{12}}}$ and $\mathbf{2}$.

Such correspondance imply different sizes of chords in musical instruments. Coincidently, piano structures that are shaped according to the size of the chords result in a shape that resembles a logarithmic function.

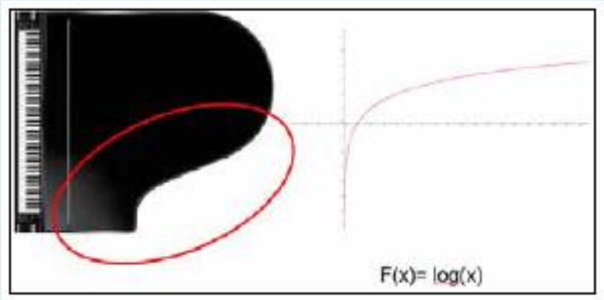

Fig. 2 – Piano and the logarithmic function
(Source: RBECM, Passo Fundo, v. 1, n. 2, jul./dec. 2018)

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Glossary

#: Symbol named "sharp" that indicates the elevation of a semitone in a note.

Diatonic Scale: a division of the octave into seven pitches.

Frequency: a physical value indicating the number of occurrences of an event in a given time span.

Fundamental Frequency: the lowest and strongest frequency of the harmonic series of a sound.

Fundamental note: the main note of a chord, from which the other chords derive from.

Harmonic: the sound of a series that constitutes a note.

Monochord: an ancient musical instrument composed of a single string over a resonance box.

Octave: the interval between musical notes with half or twice its frequency.

(Musical) Scale: an ordered sequence of tones by the vibratory frequency of sounds (usually from the lowest frequency sound to the highest frequency sound).

Semitone: interval that is half a tone and which constitutes the minimum distance in the traditional western musical system.

Tempered Scale: division of the octave into twelve equal semitones; also known as chromatic scale.

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Maths behind the Tempered Scale

As we have seen before, the tempered scale (or chromatic scale) was divided into 12 notes that can be represented by the use of logarithms. With this fact in mind, let us take a look at the shape of a logarithmic curve:

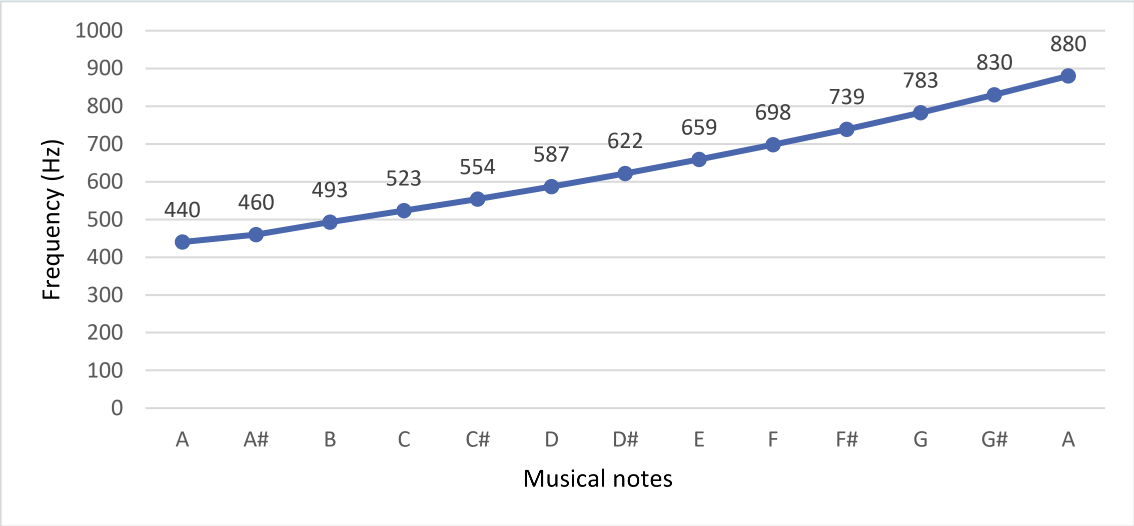

Fig. 3 – Frequencies of a tempered scale
(Source: Author; Obs: Frequency (Hz) values rounded for better displaying)

To better understand logarithms, let us look at this concept and its properties.

Logarithm of a number: logarithm function of base a

What is the logarithm of 8 with base 2?

The answer is 3, because $\mathbf{2^3 = 8}$.

The expression "3 is the logarithm of 8 with base 2" is represented as: $\mathbf{\log_2 8 = 3}$.

The logarithm of a positive number $\boldsymbol{x}$ with base $\mathbf{a}$, with $\mathbf{a} \in \mathbb{R}^+ \backslash \{\mathbf{1}\}$, to a number $\boldsymbol{y}$, such that: $\mathbf{a^y = x}$ is represented as $\mathbf{\log_a x}$, that is,

$$\mathbf{\log_a x = y \Leftrightarrow a^y = x}$$

Therefore, considering that $\mathbf{a^1 = a}$ e $\mathbf{a^0 = 1}$, then:

$\boxed{\mathbf{\log_a a = 1}}$ and $\boxed{\mathbf{\log_a 1 = 0}}$

And also:

$\boxed{\boldsymbol{\log_a a^x = x}}$ and $\boxed{\boldsymbol{a^{\log_a x} = x}}$

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Logarithm of base 10 and logarithm of base $\mathbf{e}$

Among all possible bases, two are particularly common: base 10 and base $\mathbf{e}$.

In case of a logarithm with base 10, called "common logarithm", its base can be omitted. Therefore, it can be simply represented as $\mathbf{\log x}$ instead of $\mathbf{\log_{10} x}$.

Likewise, the logarithm with base $\mathbf{e}$, called "natural logarithm", it can be represented as $\mathbf{\ln x}$ instead of $\mathbf{\log_e x}$.

The calculation of logarithms in a calculator can be done using the buttons "LOG" and "LN".

More specifically, to verify that:

- $\mathbf{10^2 = 100}$, use the button "LOG", which is used to calculate logarithms with base 10;
- $\mathbf{e^{4{,}605} \cong 100}$ use the button "LN", which is used to calculate logarithms with base $\mathbf{e}$.

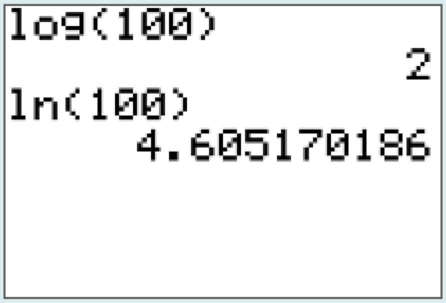

Fig. 4 – Calculation of logarithms
(Source: Graphing calculator Texas Ti-84 Plus)

Logarithmic Functions

A logarithmic function of base $\boldsymbol{a > 1}$ is a function in which

$$\mathbf{f}: \mathbb{R}^+ \rightarrow \mathbb{R}$$
$$\mathbf{x \mapsto y = \log_a x}$$

The graphs of the functions $\mathbf{f(x) = a^x}$ and $\mathbf{g(x) = \log_a x}$ are symmetrical with respect to the line of equation $\mathbf{y = x}$. Hence, the functions $\mathbf{f}$ and $\mathbf{g}$ are inverse functions.

These functions have, for $\mathbf{a > 1}$, the following graphical representations:

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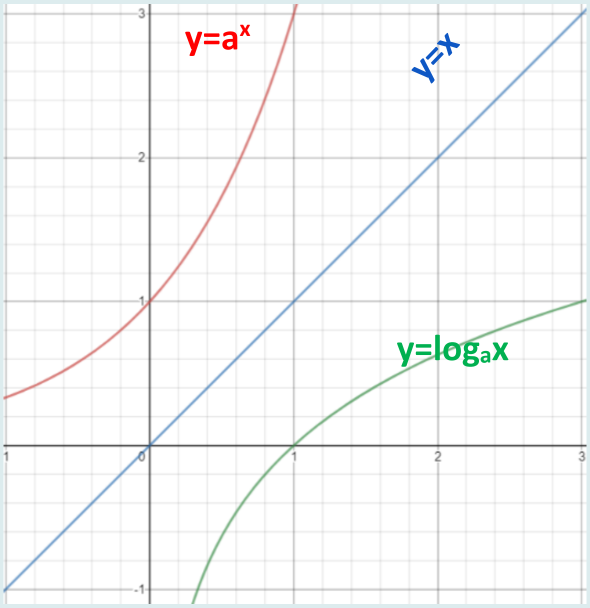

Fig. 5 – Graphs of functions a^x and $\log_a x$
(Source: Author, Desmos.com)

Properties of the logarithmic functions

The properties of the logarithmic functions are related with the properties of the respective inverse functions (exponential functions).

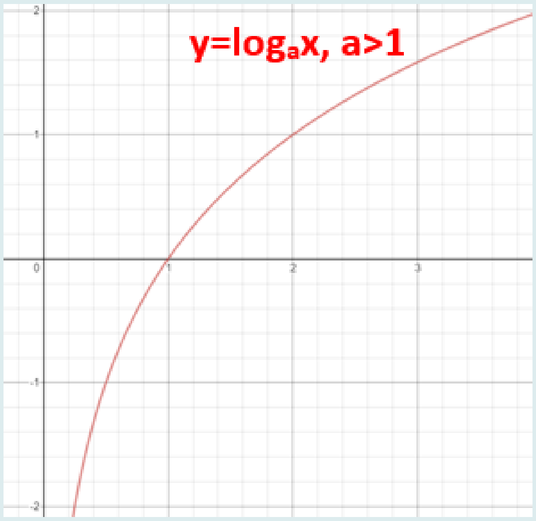

Fig. 6 - Graph of function $y=\log_a x$
(Source: Author, Desmos.com)

If $\begin{array}{c}\mathbf{f}\colon \mathbb{R}^+ \to \mathbb{R}\\ \mathbf{x} \longmapsto \mathbf{y} = \log_{\mathbf{a}} \mathbf{x}\end{array}$ with $\mathbf{a > 1}$, its graphical representation will display the shape in Fig. 6 and the properties of the function $\mathbf{f}$ will be:

- f is continuous
- Domain: $D = \mathbb{R}^+$
- Codomain: $D' = \mathbb{R}$
- $f(x) = 0 \Leftrightarrow x = 1$, that is, $f(1) = 0$
- f is strictly increasing
- $\lim\limits_{x \to +\infty} f(x) = +\infty$
- $\lim\limits_{x \to 0^+} f(x) = -\infty$, that is, the line of equation $x = 0$ is a vertical asymptote of the graphic f

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Logarithm rules of operation

The rules of operation of logarithms are related with the rules of operation of powers.

Some of the rules of operations are:

Consider $\mathbf{x} \in \mathbb{R}^+, \mathbf{y} \in \mathbb{R}^+$ and $\mathbf{a} \in \mathbb{R}^+ \setminus \{1\}$.

1. Logarithm Product Rule

The logarithm of a product is the sum of the logarithms:

$$\mathbf{log_a(xy) = log_a(x) + log_a(y)}$$

2. Logarithm Quotient Rule

The logarithm of a quotient is the difference of logarithm:

$$\mathbf{log_a\left(\frac{x}{y}\right) = log_a(x) - log_a(y)}$$

3. Logarithm Power Rule

The logarithm of a power is the product of the exponent multiplied by the logarithm of the base:

$$\mathbf{log_a(x^y) = y \times log_a(x),\ \ p \in \mathbb{R}}$$

Exceptions:

$\mathbf{log_a\left(\frac{1}{x}\right) = -log_a(x)}$ because $\frac{1}{x} = x^{-1}$ and $\mathbf{log_a(\sqrt[n]{x}) = \frac{log_a(x)}{n}}$ because $\sqrt[n]{x} = x^{\frac{1}{n}}$

4. Logarithm base change

The logarithm $\mathbf{x}$ with base $\mathbf{a}$ is the quotient of the logarithm $\mathbf{x}$ with base $\mathbf{b}$ and the logarithm of $\mathbf{a}$ with base $\mathbf{b}$:

$$\mathbf{log_a(x) = \frac{log_b(x)}{log_b(a)}}\ ,\ \ \text{with } \mathbf{a} \text{ and } \mathbf{b} \in \mathbb{R}^+ \setminus \{1\}$$

This rule is indispensable when we want to calculate the logarithm of a number whose base is different from 10 or $\mathbf{e}$, using a calculator.

Example: $\mathbf{log_3 5 = \frac{\ln 5}{\ln 3} \simeq 1,46}$ or $\mathbf{log_3 5 = \frac{\log 5}{\log 3} \simeq 1,46}$

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TASKS

TASK 1

Calculate the value of:

1.1. $\log_2 64$ 1.2. $\log_5 5$ 1.3. $\log_3\left(\frac{1}{81}\right)$ 1.4. $\log_4 1$ 1.5. $\log_{\frac{1}{4}} 2$

1.6. $\log_{\sqrt{5}} 125$ 1.7. $\log_{10} 1000$ 1.8. $\log_3\left(\frac{1}{9}\right)$ 1.9. $\log_2 \sqrt{2}$ 1.10. $\log_e \sqrt[3]{e^4}$

1.11. $\log_2\left(\frac{1}{\sqrt{2}}\right)$ 1.12. $\log_5 0,2$ 1.13. $\log_e(e^{-2}) + \log_2\left(\frac{1}{32}\right)$

TASK 2

Calculate the value of:

2.1. $\log 10000$;

2.2. $\log 0,01$;

2.3. $\ln e^{-7}$;

2.4. $\ln(\sqrt[5]{e}) - \ln(e) + \ln(e^{-3})$;

2.5. $\log(10) + \log(1) - \ln(e^2)$;

2.6. $\ln(e^{-1}) - \ln\left(e^{\frac{1}{3}}\right) + \log(\sqrt{10})$.

TASK 3

Calculate, using the properties of the logarithms, and verify the results using a calculator:

3.1. $\log_2(64 \times 16)$; 3.2. $\log_3(81:27)$; 3.3. $\log_2(32^8)$.

TASK 4

Consider that $\log_2 a = \frac{1}{5}$. Determine the value of: $\log_2\left(\frac{a^5}{8}\right)$.

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LEARN MORE…

Chromatic Scale

https://www.youtube.com/watch?v=2gy6E3X2mKQ

Introduction to Logarithms

https://www.khanacademy.org/math/algebra2/exponential-and-logarithmic-functions/introduction-to-logarithms/v/logarithms

Basic Idea and Rules for Logarithms

https://mathinsight.org/logarithm_basics

12-Tone Equal Temperament

http://www.tonalsoft.com/enc/number/12edo.aspx

Why 12 notes to the Octave?

https://www.math.uwaterloo.ca/~mrubinst/tuning/12.html

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