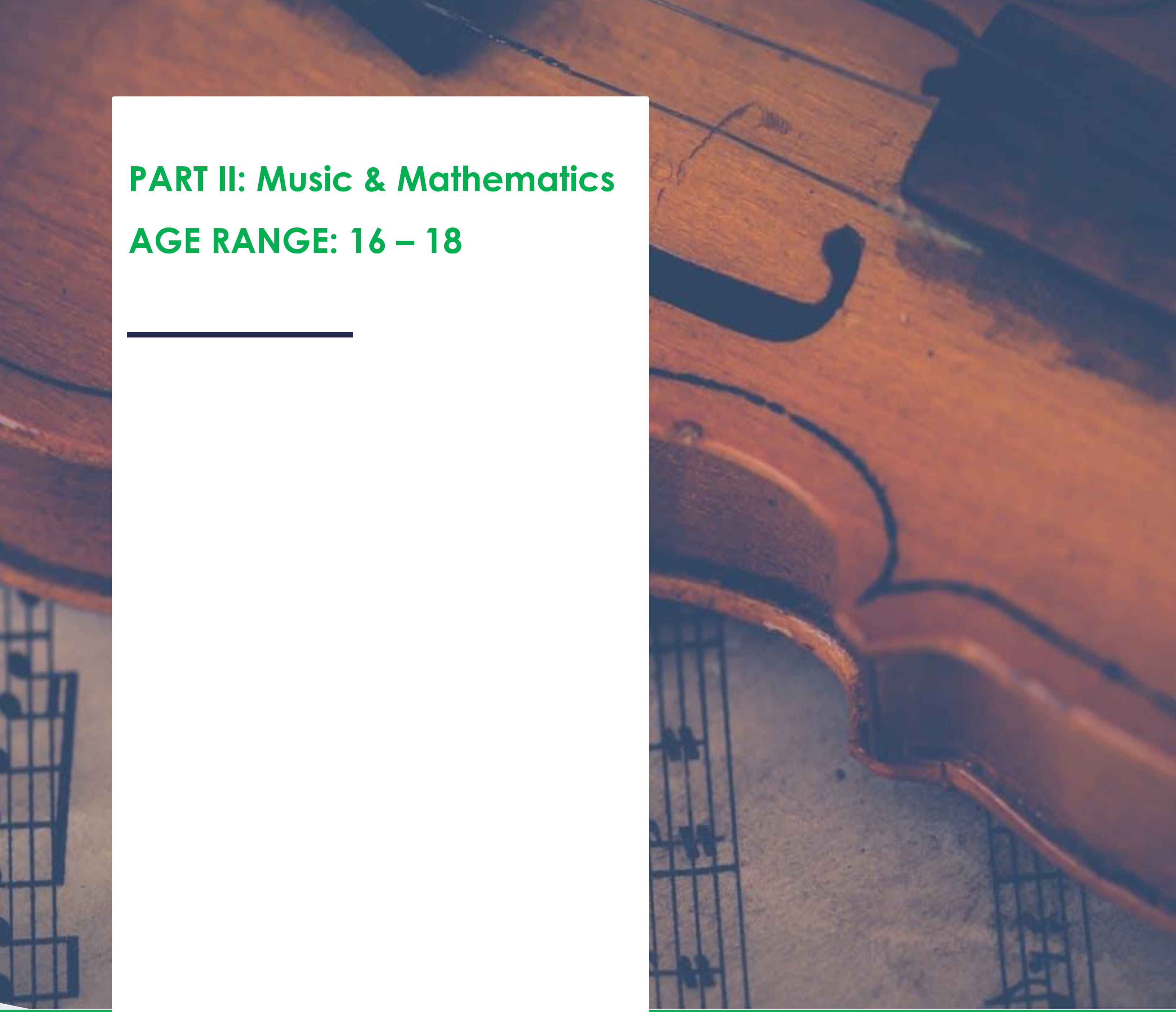
PART II: Music & Mathematics

AGE RANGE: 16 – 18



"Brown Violin"
(Source: https://www.pexels.com/photo/brown-violin-697672/)

TOOL 21: TRIGNOMETRIC FUNCTIONS IN HARMONIC SERIES

SPEL – Sociedade Promotora de Estabelecimentos de Ensino

TheArtOfMaths

Co-funded by the Erasmus+ Programme of the European Union

Educator's Guide

Title: Trigonometric Functions in Harmonic Series

Age range: 16 – 18 years old

Duration: 3 hours

Mathematical concepts: Trigonometric Functions

Artistic concepts: Harmonic series in music, musical notes, frequency of musical notes and sound waves.

General objectives: Understand the notion of trigonometric functions, calculate the period of the graph of a trigonometric function and solve trigonometric equations.

Instructions and Methodologies: It will be useful to use a graphing calculator (it can be the online graphing calculator Desmos) to show the graphs to the students and to present the solutions of the trigonometric equations. Furthermore, in order for students to get a clearer picture of the modes of vibration, please have them watch "Modes on a string" video (cf. "Learn More…") after the respective explanation.

Resources: A pen; Computer with an internet connection; Access to the website: https://www.desmos.com/

Tips for the educator: Begin by showing trigonometric functions' graphs and explain their properties. Solve a trigonometric equation for each of the three functions taught so that students can solve them by themselves.

Learning Outcomes and Competences:

At the end of this module, the student will be able to:

- Generate the graph of a trigonometric function;
- Calculate the period of a trigonometric function;
- Solve equations of the type $\sin x = a$, $\cos x = a$ and $\tan x = a$.

Debriefing and Evaluation:

Write 3 aspects you liked about this activity:	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

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Introduction

Mathematics and Music have always been connected. However, the first evidence of this relationship was only found in the sixth century BC that the first evidence of this relationship were discovered. Pythagoras compared the sound produced by hammers of different lengths, used by blacksmiths, to the sound of the monochord, of which it is believed Pythagoras was the inventor.

This comparison allowed Pythagoras to discover and improve the mathematical reasons behind the sounds through the study of the sounds produced by the monochord. He divided the string into two equal parts, then into three equal parts, and so on. He matched the sounds mathematically according to the subdivisions he was making and created the Pythagorean scale, in which each note maintained a well-defined relationship with the other.

Many people and cultures have created their own scales. One example was the Chinese people who has created the pentatonic scale. Western culture, however, adopted a 12-tone equal temperament, known as a tempered scale or chromatic scale.

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Harmonic Series

It is of general knowledge that the natural musical notes are A, B, C, D, E, F and G. Nevertheless, these are represented in most countries by the solfège naming convention Do–Re–Mi–Fa–Sol–La–Ti (or Si) in accordance to the following correspondence: C-Do, D-Re, E-Mi, F-Fa, G-Sol, A-La and B-Ti (or Si). The definition of these notes was widely influenced by Mathematics.

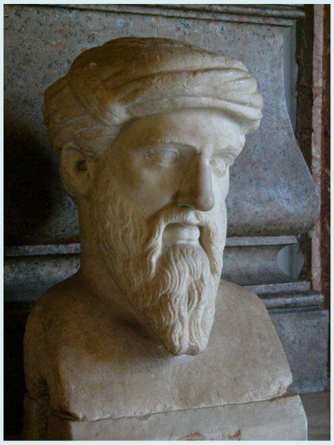

Fig. 1 – Pythagoras Bust
(Source:https://commons.wikimedia.org/wiki/File:Kapitolinischer_Pythagoras_adjusted.jpg)

In the sixth century BC, Pythagoras realized that when vibrating a string it not only vibrated in its full extent, but it also formed a series of nodes, which divide into smaller sections, the partials, which vibrate at frequencies higher than the fundamental.

To study the relationship between the length of the vibrating string and the musical tone produced by it, he used a monochord.

Figure 2 shows the nodes and partials of the first four frequencies of a series. For an easy understanding, they are shown separately, but on a real string, all overlap, generating a complex design, similar to the waveform of the instrument.

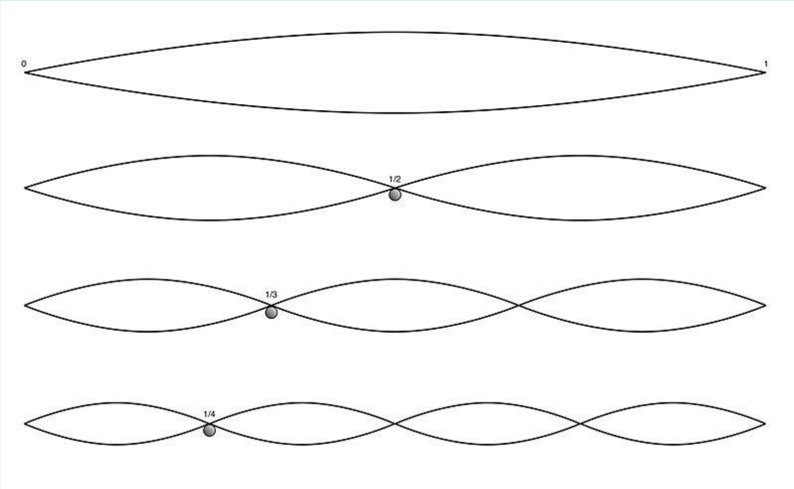

Fig. 2 – Modes of vibration of the first 4 harmonics
((Source:https://pt.wikipedia.org/wiki/Frequ%C3%AAncia_fundamental#/media/Ficheiro:Overtone.jpg)

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Imagine a string stretched out, stuck at its ends. When we touch one end of this string, it vibrates (note the first drawing in Figure 3) and produces a note that is called a fundamental note.

Fig. 3 – Modes of vibration of a fundamental note 1(f)
(Source:https://pt.wikipedia.org/wiki/Frequ%C3%AAncia_fundamental#/media/Ficheiro:Overtone.jpg)

Pythagoras decided to divide a string into two parts (Figure 4) by touching it in the middle. The sound produced was exactly the same, but with a higher frequency (usually expressed as "same note, an octave higher"). It has since been proved that whenever the number of divisions (or the harmonic number) is a multiple of an earlier number, then the sound will be repeated but with a higher pitch.

Fig. 4 – Modes of vibration of a fundamental note 2(f)
(Source:https://pt.wikipedia.org/wiki/Frequ%C3%AAncia_fundamental#/media/Ficheiro:Overtone.jpg)

He then decided to try out what it would sound like if the string was divided into 3 parts (Figure 5) and noticed that a new sound, different from the previous one, came out. This time, it was not the "same note, an octave higher", but a completely different note, which deserved a different name - the fifth.

Fig. 5 – Modes of vibration of a fundamental note 3(f)
(Source:https://pt.wikipedia.org/wiki/Frequ%C3%AAncia_fundamental#/media/Ficheiro:Overtone.jpg)

This sound, although different, matched well with the previous sound. It created a pleasant harmony to the ear, which had to do with the fact that the divisions done had the mathematical relations of 1/2 and 2/3. With the division of the string into four parts, he obtained the note, now known as "fourth". These three notes are in consonance with the fundamental note.

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Thus, he continued to subdivide the string, obtaining the harmonics of the fundamental note, and, by mathematically combining the sounds, he created scales which result in notes that naturally related to each other. Over time, the notes have been given the names we know of today, which were mentioned earlier.

In this process, each note coming from an object, suffers the influence of the fundamental frequency that excites other harmonics, which results in a series of frequencies - the harmonic series. The harmonic series are infinite series, composed of sinusoidal waves with all the integer multiple frequencies of the fundamental frequency. There is not a single harmonic series, but rather a different series for each fundamental frequency.

Let us look at an example of a harmonic series that starts at A_2 / $Lá_1$ (110 Hz). The first 16 harmonics for that series can be observed in the following table:

Harmonic #	Note (English)	Note (Neo-latin)	Frequency (Hz)
1 (F)	A_2	$Lá_1$	110
2	A_3	$Lá_2$	220
3	E_4	Mi_3	330
4	A_5	$Lá_3$	440
5	$C\#_5$	$Do\#_4$	550
6	E_4	Mi_4	660
7	G_4	Sol_4	770
8	A_5	$Lá_4$	880
9	B_5	Si_4	990
10	$C\#_6$	$Do\#_5$	1100
11	$D\#_6$	$Ré\#_5$	1210
12	E_6	Mi_5	1320
13	$F\#_6$	$Fá\#_5$	1430
14	G_6	Sol_5	1540
15	$G\#_5$	$Sol\#_5$	1650
16	A_6	$Lá_5$	1760

Table 1 – First 16 harmonics

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Sound Waves

When a musical instrument produces a sound, it vibrates, and a series of sinusoidal waves are emitted. In addition to the fundamental frequency that defines the note, several harmonic frequencies (wave with a frequency that is a positive integer multiple of the frequency of the original wave) are also emitted. This way, the existence of several frequencies in the same time interval, produced by the same sound source, leads to the formation of complex/irregular waves, resulting from the sum of simple sinusoidal harmonics, as shown in Figure 6.

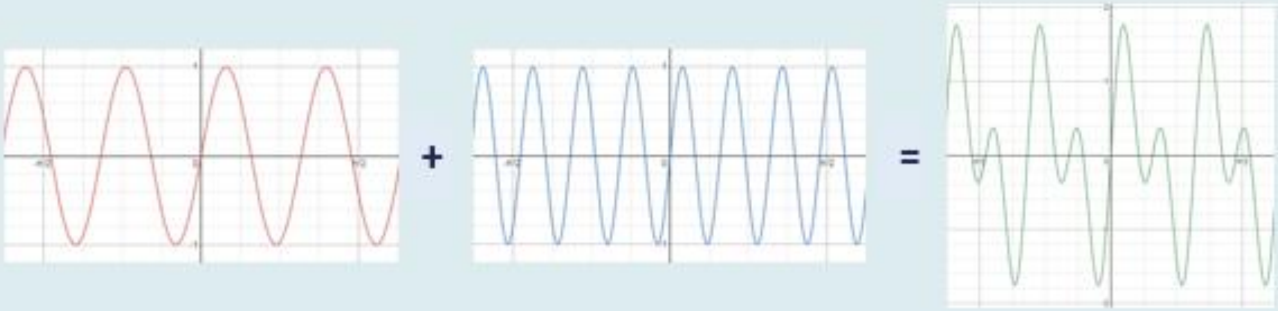

Fig. 6 – Formation of an irregular sound wave
(Source: Author, at Desmos)

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Glossary

Fifth: interval between a musical note and another, which is four degrees away from the first, within a scale.

Fourth: interval between one musical note and another, which is three degrees away from the first, within a scale.

Frequency: physical quantity indicating the number of occurrences of an event in a given time span.

Fundamental Frequency: the lowest and strongest component frequency of the harmonic series of a sound.

Fundamental note: main note of a chord, from which the other chords derive from

Harmonic Series: set of waves composed by the fundamental frequency and of all the integer multiples of this frequency.

Harmonic: sound of a series that constitutes a note.

Harmony: simultaneous combination of sounds.

Monochord: an old musical instrument composed of a resonance box, on which was extended a single string fastened by two mobile supports.

Octave: interval between a musical note and another one with half or twice its frequency.

Pentatonic scale: set of all scales consisting of five notes or tones.

Pitch: high frequency sound from human hearing, usually above 5 KHz.

Sinusoidal wave: a mathematical curve that describes a smooth periodic oscillation

(Musical) Scale: ordered sequence of tones by the vibratory frequency of sounds (usually from the lowest frequency sound to the highest frequency sound).

Tempered Scale: division of the octave into twelve equal semitones.

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Math behind the Harmonic Series

When a musical instrument is able to produce sounds, it vibrates and a series of sinusoidal waves are emitted. When isolated, these waves obey the following mathematical function: $\mathbf{f(x) = \sin(f_i.\,2\pi x)}$, where $\mathbf{f_i}$ is the frequency of the harmonic of order $\mathbf{i}$.

Let us look at an example: if the frequency of a harmonic is 1, then the waves emitted will look like this:

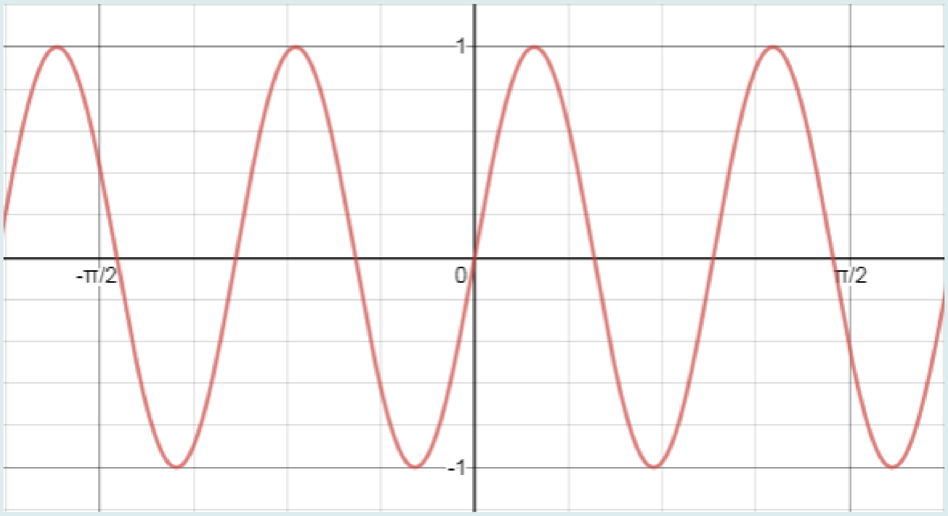

Fig. 7 – Sinusoidal wave of the function $\mathbf{f(x) = \sin(f_i.\,2\pi x)}$ where $\mathbf{f_i = 1}$.
(Source: Author, at Desmos.com)

Let us spend some time with trigonometry and trigonometric functions to better understand sinusoidal waves.

1. Trigonometric Functions

Trigonometric functions as real functions of real variable

If to any real number $\mathbf{x}$ matches one and only one real number $\mathbf{y}$ so that $\mathbf{y = \sin x}$ and $\mathbf{y = \cos x}$, then $\mathbf{y = \sin x}$, $\mathbf{y = \cos x}$ and $\mathbf{y = \tan x}$ ($\mathbf{\tan x = \frac{\sin x}{\cos x}}$) are now considered real functions of real variable.

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Domain, codomain, extrema and zeros of trigonometric functions

In a function $\mathbf{f(x)}$, the x value is any real number and is usually called **domain.** As for the set Y into which all of the output of the function is constrained to fall is called **codomain.** In other words, any value that goes into a function is the domain and the value that come out is the codomain.

When modelling a function $\mathbf{f(x)}$, you will notice that it has a largest and a smallest value. These are the **extrema** and correspond to the **maximum and minimum point** in a function. Additionally, a function may have zeros. These are the intersections in the x-axis, that is, the zero of a function is an input value that produces an output of 0.

Consider the graphs of the functions: $\mathbf{y = \sin x}$, $\mathbf{y = \cos x}$ and $\mathbf{y = \tan x}$ in the interval $[\mathbf{-2\pi, 2\pi}]$.

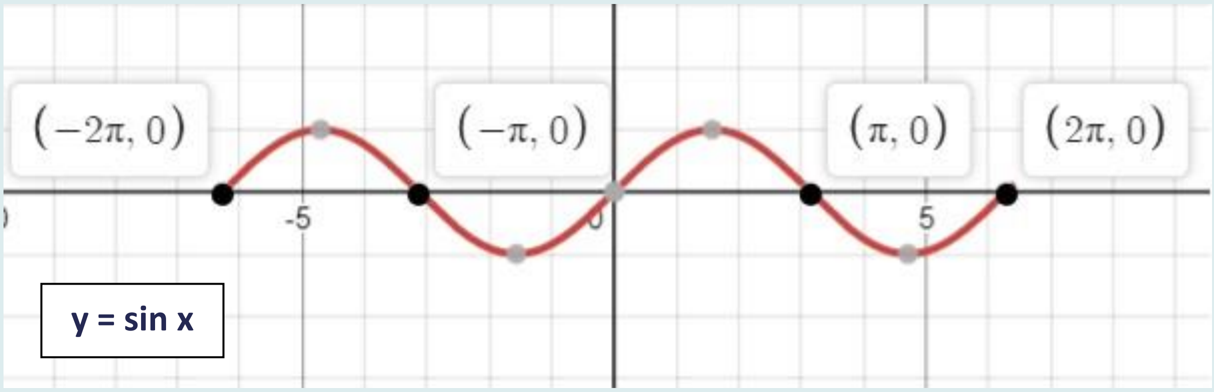

Fig. 8 – Graphical function of y=sin x
(Source: Author, at Desmos.com)

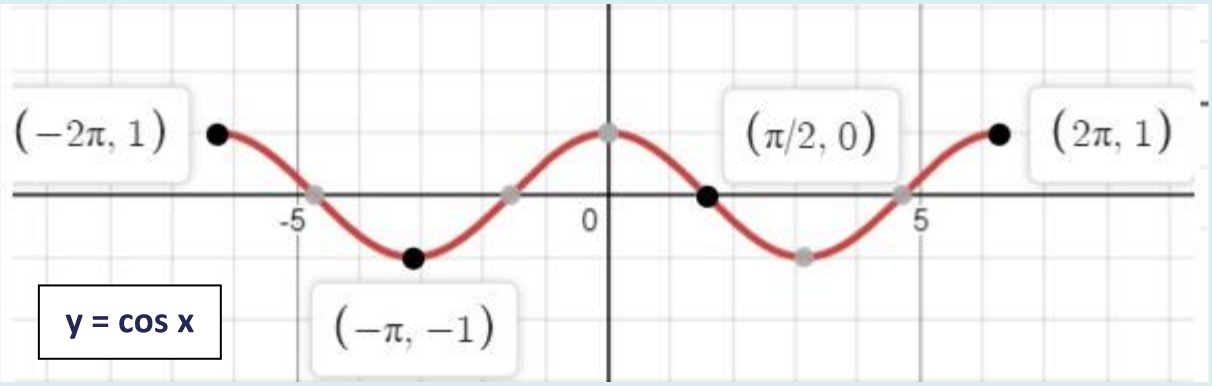

Fig. 9 – Graphical function of y=cos x
(Source: Author, at Desmos.com)

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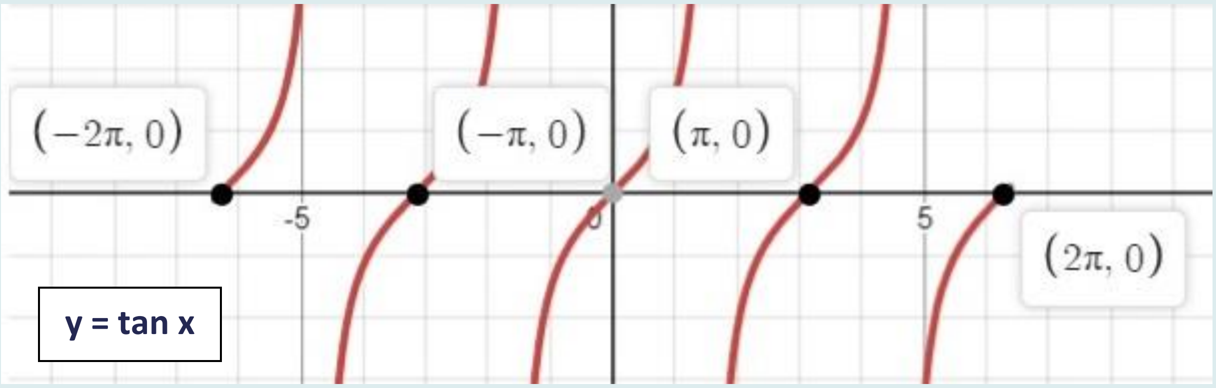

Fig. 10 – Graphical function of y=tan x
(Source: Author, at Desmos.com)

By observing the graphs, it is possible to conclude that:

Function	$y = \sin x$	$y = \cos x$	$y = \tan x$
Domain	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}\backslash\left\{\frac{\pi}{2} + k\pi, k\epsilon\mathbb{Z}\right\}$
Codomain	[-1 , 1]	[-1 , 1]	$\mathbb{R}$
Maximum point	1 to: $x = \frac{\pi}{2} + 2k\pi, k \epsilon \mathbb{Z}$	1 to: $x = 2k\pi, k \epsilon \mathbb{Z}$	------
Minimum point	-1 to: $x = \frac{3\pi}{2} + 2k\pi, k\epsilon\mathbb{Z}$	-1 to: $x = \pi + 2k\pi, k\epsilon\mathbb{Z}$	------
Zeros	$x = k\pi, k\epsilon\mathbb{Z}$	$x = \frac{\pi}{2} + k\pi, k\epsilon\mathbb{Z}$	$x = k\pi, k\epsilon\mathbb{Z}$

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2. Monotony of trigonometric functions

Looking at the previous graphs in the interval $[-2\pi, 2\pi]$, is it possible to conclude that:

- $\sin(x)$ is increasing in $\left[\frac{3\pi}{2}, 2\pi\right]$, and decreasing in $\left[\frac{\pi}{2}, \pi\right]$ and in $\left[\pi, \frac{3\pi}{2}\right]$;
- $\cos(x)$ is increasing in $\left[\pi, \frac{3\pi}{2}\right]$ and in $\left[\frac{3\pi}{2}, 2\pi\right]$, and decreasing in $\left[0, \frac{\pi}{2}\right]$ and in $\left[\frac{\pi}{2}, \pi\right]$;

Regarding the function $y = \tan x$, it is possible to conclude that the function is increasing at all intervals in which it is defined.

3. Symmetry and parity of trigonometric functions

Even function

- A function f is even if, and only if $f(-x) = f(x)$, $\forall x \in D_f$.
- The graph of an even function is symmetric with respect to the y-axis.

The function $y = \cos x$ is an **even function**, that is, $\cos(-x) = \cos x$, $\forall x \in \mathbb{R}$.

Odd function

- A function f is odd if, and only if $f(-x) = -f(x)$, $\forall x \in D_f$.
- The graph of an odd function is symmetric with respect to the origin of the coordinates.

The function $y = \sin x$ is an **odd function**, that is, $\sin(-x) = -\sin x$, $\forall x \in \mathbb{R}$.

The function $y = \tan x$ is an **odd function**, that is, $\tan(-x) = -\tan x$, $\forall x \in \mathbb{R}\backslash\left\{\frac{\pi}{2} + k\pi, k\epsilon\mathbb{Z}\right\}$.

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4. Period of the trigonometric functions

The function $\mathbf{f}$ is periodic of **period** $\mathbf{p}$ if $\mathbf{p}$ is the smallest positive constant, such that $\mathbf{f(x+p)=f(x)}$ for all the $\mathbf{x}$ of the $\mathbf{f}$ domain.

- The period of the function $y = \sin x$ is 2π: $\sin(x + k \times 2\pi) = \sin x, \forall x \in \mathbb{R}\ (k \in \mathbb{Z})$;
- The period of the function $y = \cos x$ is 2π: $\cos(x + k \times 2\pi) = \cos x, \forall x \in \mathbb{R}\ (k \in \mathbb{Z})$;
- The period of the function $y = \tan x$ is π: $\tan(x + k\pi) = \tan x, \forall x \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi\right\}\ (k \in \mathbb{Z})$.

In general, to $\mathbf{k \neq 0}$:

Function	$\mathbf{y=A+B\sin(kx+C)}$	$\mathbf{y=A+B\cos(kx+C)}$	$\mathbf{y=A+B\tan(kx+C)}$
Period	$\mathbf{\frac{2\pi}{\lvert k\rvert}}$	$\mathbf{\frac{2\pi}{\lvert k\rvert}}$	$\mathbf{\frac{\pi}{\lvert k\rvert}}$

5. Solving equations of the type $\sin x = a$

In general, in order to solve, in $\mathbb{R}$, an equation of the type $\mathbf{\sin x = a}$ we must consider the following information:

- The equation of the type $\sin x = a$ has only a solution if $a \in [-1,1]$.
- In the interval $[0,2\pi]$ there are two values that have same sine value: α and $\pi - \alpha$.
- $\sin x = \sin\alpha \Leftrightarrow x = \alpha + 2k\pi \vee x = \pi - \alpha + 2k\pi, k \in \mathbb{Z}$.

Example of the resolution of an equation of the type $\mathbf{\sin x = a}$:

Solve, in $\mathbb{R}$, the equation $\mathbf{\sin x = -\frac{\sqrt{2}}{2}}$.

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➔ 1st step:

Find a solution of the equation $\mathbf{\sin x = -\frac{\sqrt{2}}{2}}$.

We know that $\mathbf{\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}}$ and that $\mathbf{\sin(-x) = -\sin x}$. Therefore, $\mathbf{\sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}}$.

A solution of the equation is $\mathbf{-\frac{\pi}{4}}$.

➔ 2nd step:

Calculate $\mathbf{\pi - \frac{\pi}{4} = \pi + \frac{\pi}{4} = \frac{5\pi}{4}}$.

➔ 3rd step:

Write the general solution of the equation: $\mathbf{x = -\frac{\pi}{4} + 2k\pi \vee x = \frac{5\pi}{4} + 2k\pi, k \in \mathbb{Z}}$.

By using the online graphic calculator Desmos, we can confirm the two obtained values:

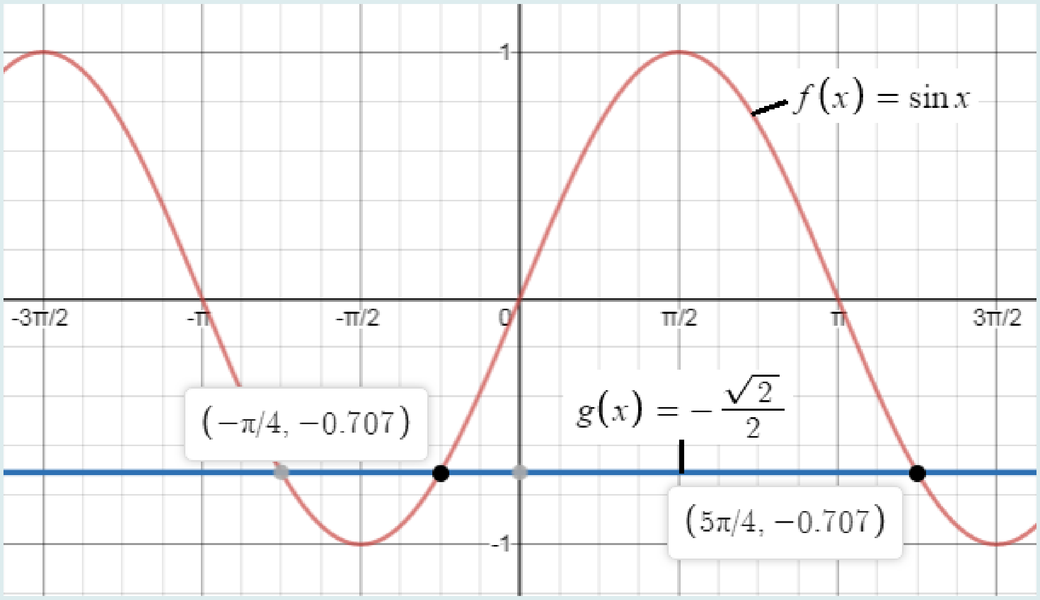

Fig. 11 – Graphical representation of the equation $\sin x = -\frac{\sqrt{2}}{2}$, in $\mathbb{R}$.

(Source: Author, at Desmos)

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6. Solving equations of type $\cos x = a$

After having assimilated the resolution of an equation of the type $\mathbf{\sin x = a}$, it is very simple to solve an equation of the type $\mathbf{\cos x = a}$.

The difference is only in observing that: $\boldsymbol{\alpha}$ and $\boldsymbol{-\alpha}$ have the same cosine, that is, $\mathbf{\cos(\alpha) = \cos(-\alpha) = a}$.

Therefore, to solve, in $\mathbb{R}$, an equation of type $\mathbf{\cos x = a}$ we must consider the following information:

- The equation of type $\cos x = a$ only has a solution if a $\in [-1,1]$.
- In the interval [0, 2π] there are two values that have the same sine: α and $-\alpha$.
- $\cos x = \cos \alpha \Leftrightarrow x = \alpha + 2k\pi \vee x = -\alpha + 2k\pi, k \in \mathbb{Z}$.

Example of a resolution of an equation of type $\mathbf{\cos x = a}$.

Solve, in $\mathbb{R}$, the equation $\mathbf{1 - 2\cos x = 0}$.

Resolution: $\mathbf{1 - 2\cos x = 0 \Leftrightarrow -2\cos x = -1 \Leftrightarrow \cos x = \frac{1}{2}}$.

➔ 1st step:

Determine $\boldsymbol{\alpha}$, in radians, so that $\mathbf{\cos \alpha = \frac{1}{2}}$.

We know that $\mathbf{\cos \frac{\pi}{3} = \frac{1}{2}}$, therefore, $= \mathbf{\frac{\pi}{3}}$.

➔ 2nd step:

If $\boldsymbol{\alpha = \frac{\pi}{3}}$ is a solution of the equation, so is $\boldsymbol{-\alpha = -\frac{\pi}{3}}$.

➔ 3rd step:

Write the general solution of the equation: $\mathbf{x = \frac{\pi}{3} + 2k\pi \vee x = -\frac{\pi}{3} + 2k\pi, k \in \mathbb{Z}}$.

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By using the online graphic calculator Desmos, we can confirm the two obtained values:

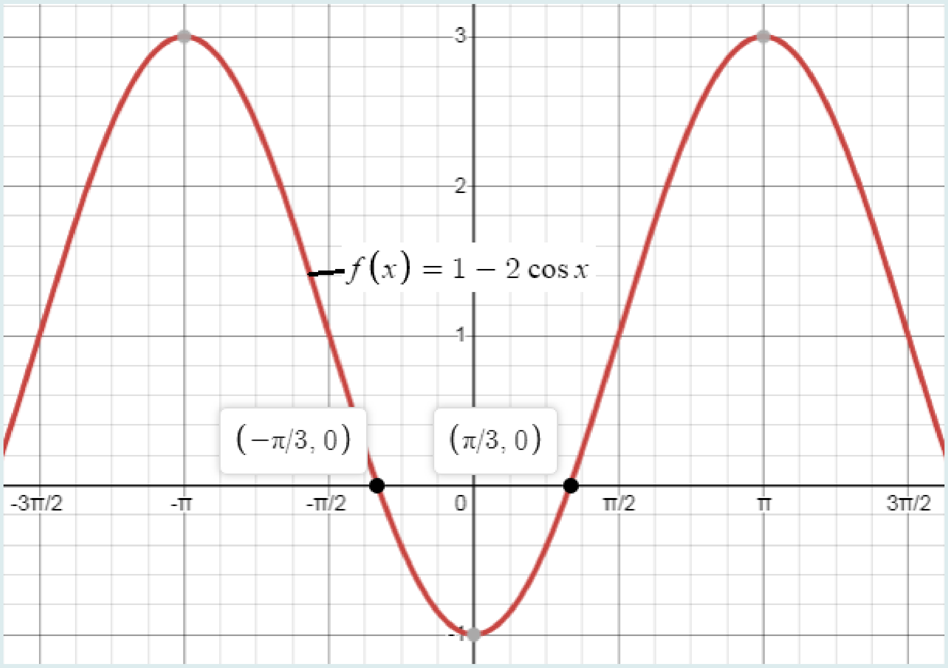

Fig. 12 – Graphical representation of the equation $1 - 2cos\ x = 0$
(Source: Author, at Desmos)

7. Solving equations of type $\tan x = a$

In the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$, the equation $\mathbf{\tan x = a}$ has one and only solution: let it be $\boldsymbol{\alpha}$.

Since the period of the function $\mathbf{y = \tan x}$ is $\boldsymbol{\pi}$, we conclude that if $\boldsymbol{\alpha}$ is the solution of the equation $\mathbf{\tan x = a}$, then $\boldsymbol{\alpha + k\pi, k} \in \mathbb{Z}$ is also a solution.

Therefore, to solve, in $\mathbb{R}$ an equation of the type $\mathbf{\tan x = a}$ we must consider the following information:

- The equation $\tan x = a$ has solution for any real value of a.
- $\tan x = \tan \alpha \Leftrightarrow x = \alpha + k\pi, k \in \mathbb{Z}$.

Example of a resolution of an equation of type $\mathbf{\tan x = a}$

Solve the equation $\mathbf{\tan x = \sqrt{3},\ 0 \leq x \leq 2\pi\ (rad)}$.

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It is known that $\mathbf{tan}\,\frac{\pi}{3} = \sqrt{3}$. Therefore, $\mathbf{x} = \frac{\pi}{3} + \mathbf{k\pi}, \mathbf{k} \in \mathbb{Z}$.

$$\mathbf{k = 0} \;\Longrightarrow \mathbf{x} = \frac{\pi}{3};$$

$$\mathbf{k = 1} \;\Longrightarrow \mathbf{x} = \frac{4\pi}{3};$$

$$\cancel{\mathbf{k}} = \mathbf{2} \;\Longrightarrow \mathbf{x} = \frac{\pi}{3} + \mathbf{2\pi} \;\; (\textbf{greater than } \mathbf{2\pi});$$

$$\cancel{\mathbf{k}} = \mathbf{-1} \;\Longrightarrow \mathbf{x} = \frac{\pi}{3} - \mathbf{2\pi} \;\; (\textbf{less than } \mathbf{0}).$$

Therefore, $\mathbf{S} = \left\{\frac{\pi}{3}, \frac{4\pi}{3}\right\}$.

By using the online graphic calculator Desmos, we can confirm the two obtained values:

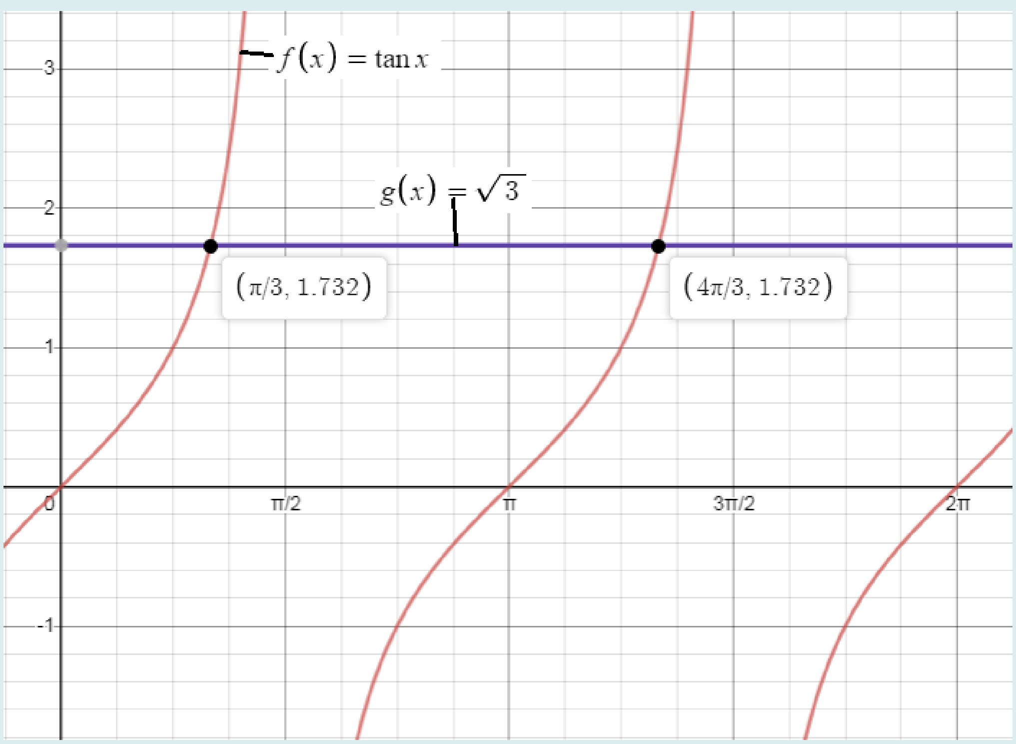

Fig. 13 – Graphical representation of the equation $\tan x = \sqrt{3},\ 0 \le x \le 2\pi$ (rad).
(Source: Author, at Desmos)

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TASKS

TASK 1

Determine the period of each of the following trigonometric functions:

1.1. $y = \sin(2x)$;

1.2. $y = 5\sin\left(\frac{\pi}{3}x\right)$;

1.3. $y = -2\cos(-5x)$;

1.4. $y = -20\cos(\pi x)$;

1.5. $y = -3\tan(2x)$;

1.6. $y = -3\tan\left(-\frac{\pi}{2}x\right)$.

TASK 2

Solve, in $\mathbb{R}$, the following trigonometric equations:

2.1. $-2\sin(x) = \sqrt{2}$;

2.2. $2\sin(x) + \sqrt{3} = 0$;

2.3. $-2\sin(x) = -4$;

2.4. $2\sin(2x) - 1 = 0$.

TASK 3

Solve each of the following equations in the indicated sets.

Note: Show the solutions in radians.

3.1. $\cos(x) = -\frac{\sqrt{2}}{2}$, in $\mathbb{R}$;

3.2. $2\cos(x) + 1 = 0$, in $\mathbb{R}$;

3.3. $\cos(x) = -1$, in $[0, 3\pi]$.

TASK 4

Solve each of the following equations in the indicated sets.

Note: Show the solutions in radians.

4.1. $3\tan\left(\frac{x}{2}\right) = -\sqrt{3}$, in $\mathbb{R}$;

4.2. $\tan(2x) = 1$, in $[0, 2\pi]$.

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https://www.oberton.org/en/overtone-singing/harmonic-series/

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http://in.music.sc.edu/fs/bain/atmi02/hs/hs.pdf

Trignometric functions

https://www.khanacademy.org/math/algebra-home/alg-trig-functions

Explore graphs of trigonometric functions with Desmos web application

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