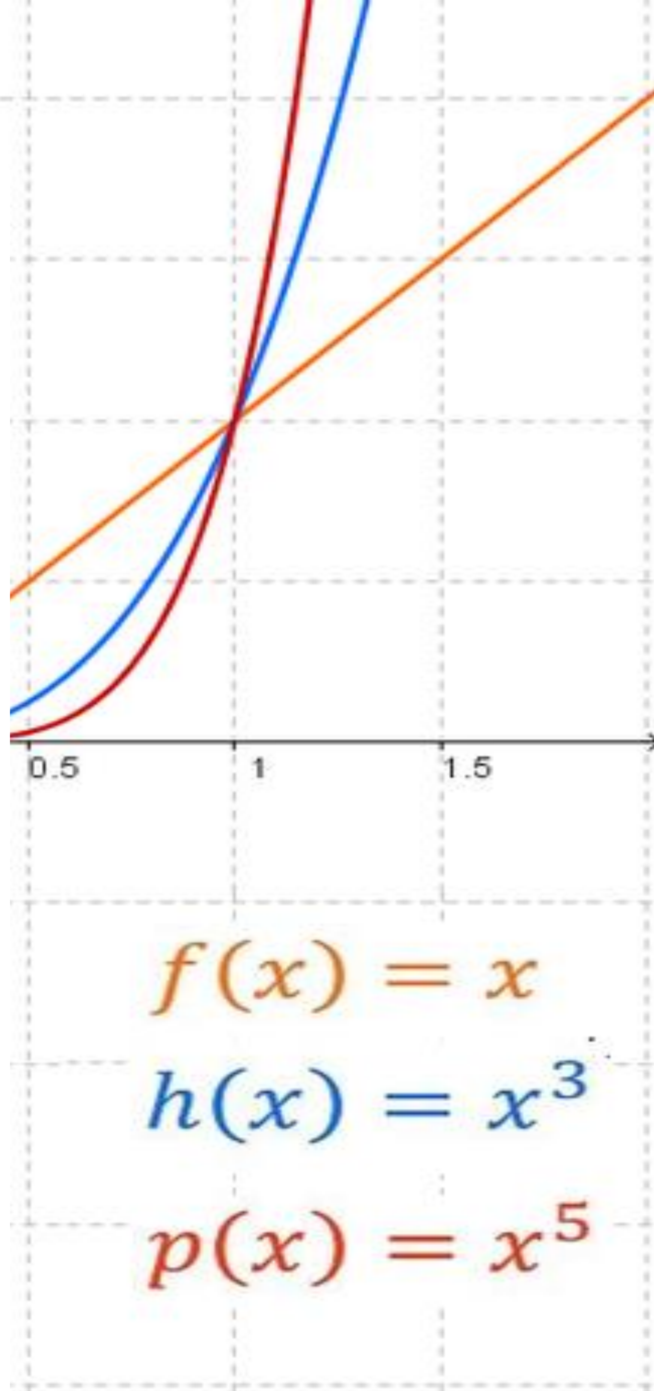
PART II: Music & Mathematics

AGE RANGE: 13 – 15



Power function graph degree 1 3 5
(Source: J Hokkanen from Wiki Commons)

TOOL 18: POWERS IN THE TEMPERED SCALE

SPEL – Sociedade Promotora de Estabelecimentos de Ensino

TheArtOf Maths

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Educator's Guide

Title: Powers in the Tempered Scale

Age range: 13 – 15 years old

Duration: 3 hours

Mathematical concepts: Powers, Properties of powers, Operation with powers

Artistic concepts: 12-tone equal temperament, musical notes and frequencies of musical notes

General objectives: To understand the concept of power, its properties, and to perform operations involving powers

Instructions and Methodologies: It will be useful to use a scientific calculator (it can be the online graphing calculator Desmos), so the student can learn how to calculate exponents in a calculator and be able to verify the tasks' results/solutions

Resources: Computer with an internet connection; Access to the website: https://www.desmos.com/

Tips for the educator: Begin by giving one or two examples of each operational rule with increasing difficulty to show how to proceed, so that students can then solve tasks on their own.

Learning Outcomes and Competences:

At the end of this tool, the student will be able to:

- Dominate the rules of powers;
- Calculate the value of numerical expressions by using the rules of powers.

Debriefing and Evaluation:

Write 3 aspects you liked about this activity	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

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Introduction

Mathematics and Music have always been connected. However, the first evidence of this relationship was only found in the sixth century BC that the first evidence of this relationship were discovered. Pythagoras compared the sound produced by hammers of different lengths, used by blacksmiths, to the sound of the monochord, of which it is believed Pythagoras was the inventor.

This comparison allowed Pythagoras to discover and improve the mathematical reasons behind the sounds through the study of the sounds produced by the monochord. He divided the string into two equal parts, then into three equal parts, and so on. He matched the sounds mathematically according to the subdivisions he was making and created the Pythagorean scale, in which each note maintained a well-defined relationship with the other.

The Pythagorean scale is the basis of the diatonic scale, consisting of seven notes, which is the basis of the formation of all the other scales used in Western music. One of the scales that emerged in Western culture was the 12-tone equal temperament, known as tempered scale or chromatic scale, in which there is a greater consonance between notes.

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Powers in the Tempered Scale

In the sixth century BC, Pythagoras used the monochord to study the relationship between the length of the vibrating string and the musical tone it produces. Imagine a string stretched and fixed both ends. When an end of this string is touched, it vibrates and produces a note that is called a fundamental note. Pythagoras then divided the string into two equal parts, then into three and so on. As he continued to subdivide the string, obtaining the harmonics of the fundamental note, and, by mathematically combining the sounds, he created scales which resulted in notes that were naturally related to each other.

By keeping the same intervals (numerical ratio of $\frac{3}{2}$) between the notes and by starting from the octave interval given by the frequencies $\mathbf{f_0}$and $\mathbf{2f_0}$, the Pythagorean diatonic scale can be formed. The notes obtained, commonly known as C, R, E, F, G, A and B, otherwise represented in most countries by the solfège naming convention Do-Re–Mi–Fa–Sol–La–Ti (or Si) in accordance to the correspondence C-Do, D-Re, E-Mi, F-Fa, G-Sol, A-La and B-Ti (or Si), form the so-called diatonic scale of seven notes that for centuries was the basis of other scales.

From the Middle Ages onwards, it was noticeable that some notes were too close to each other (for example, the notes B and C) so a decision has been made to create a scale in which the frequency interval between all notes would be the same ratio. The value of it being the interval between the notes C and the B (a semitone). As a result, the 12-tone equal temperament was formed and improved by J. S. Bach.

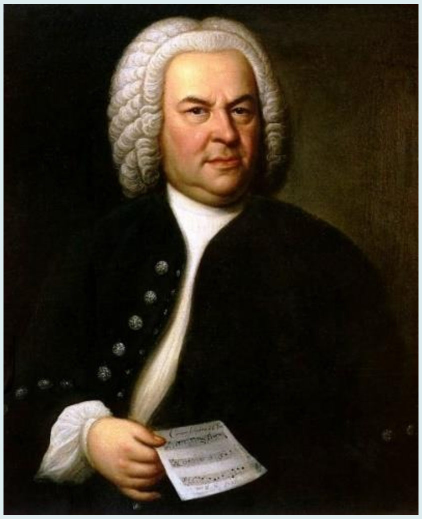

Fig. 1 - Johann Sebastian Bach
(Source:https://commons.wikimedia.org/wiki/Johann_Sebastian_Bach)

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Unlike Pythagoras who had formed the diatonic scale by obtaining 7 notes through a division that can be represented by fractions, this new temperament can be represented by powers of 2 and resulted in 12 notes: C, C#, D, D#, E, F, F#, G, G#, A, A# and B.

Note	Semitones	Equal Temperament scale
C	0	$\mathbf{2^0 = 1}$
C#	1	$\mathbf{2^{1/12}}$
D	2	$\mathbf{2^{2/12}}$
D#	3	$\mathbf{2^{3/12}}$
E	4	$\mathbf{2^{4/12}}$
F	5	$\mathbf{2^{5/12}}$
F#	6	$\mathbf{2^{6/12}}$
G	7	$\mathbf{2^{7/12}}$
G#	8	$\mathbf{2^{8/12}}$
A	9	$\mathbf{2^{9/12}}$
A#	10	$\mathbf{2^{10/12}}$
B	11	$\mathbf{2^{11/12}}$
C (octave)	12	$\mathbf{2^1 = 2}$

Table 1: Octave division in equal temperament

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Glossary

#: Symbol named "sharp" that indicates the elevation of a semitone in a note.

Diatonic Scale: a division of the octave into seven pitches.

Frequency: a physical value indicating the number of occurrences of an event in a given time span.

Fundamental Frequency: the lowest and strongest frequency of the harmonic series of a sound.

Fundamental note: the main note of a chord, from which the other chords derive from.

Harmonic: the sound of a series that constitutes a note.

Monochord: an ancient musical instrument composed of a single string over a resonance box.

Octave: the interval between musical notes with half or twice its frequency.

(Musical) Scale: an ordered sequence of tones by the vibratory frequency of sounds (usually from the lowest frequency sound to the highest frequency sound).

Semitone: interval that is half a tone and which constitutes the minimum distance in the traditional western musical system.

Tempered Scale: division of the octave into twelve equal semitones; also known as chromatic scale.

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Maths behind the Tempered Scale

As we have seen before, the chromatic or tempered scale was divided into 12 notes. The pitch of the next note is obtained by multiplying the pitch value of the previous note by powers of 2, that is $\mathbf{2^{1/12}}$.

Example: $\mathbf{Tempered\ tuning\ of\ note\ D} = \mathbf{2^{1/12}} \times \mathbf{Tempered\ tuning\ of\ note\ c\#} = \mathbf{2^{1/12}} \times \mathbf{2^{1/12}} = \mathbf{2^{2/12}}$

To better understand powers, let us look at this concept and its properties.

Powers

In a multiplication, when the factors are equal, we can represent them in the form of power.

For example: $\mathbf{2 \times 2 \times 2 = 2^3}$

Base: 2 (the repeating factor);

Exponent: 3 (the number of times the factor is repeated).

Properties of Powers

The **product of powers with the same base** results in a power with the same base and an exponent equal to the sum of the exponents.

$$\mathbf{a^m \times a^n = a^{m+n}; a \in \mathbb{R}; m \in \mathbb{Z}} \text{ and } \mathbf{n \in \mathbb{Z}}$$

Examples: $\mathbf{3^2 \times 3^4 = 3^{2+4} = 3^6}$ and $\left(\frac{2}{3}\right)^3 \times \left(\frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)^{3+2} = \left(\frac{2}{3}\right)^5$

The **product of powers with the same exponent** results in a power with the same exponent and a base equal to the multiplication of the bases.

$$\mathbf{a^m \times b^m = (a\ x\ b)^m; a \in \mathbb{R}; b \in \mathbb{R}} \text{ and } \mathbf{m \in \mathbb{Z}}$$

Examples: $\mathbf{3^2 \times 4^2 = (3 \times 4)^2 = 12^2}$ and $\left(\frac{3}{5}\right)^2 \times \left(\frac{4}{5}\right)^2 = \left(\frac{3}{5} \times \frac{4}{5}\right)^2 = \left(\frac{12}{25}\right)^2$

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The **quotient of powers with the same base** results in a power with the same base and with the exponent equal to the difference between exponents.

$$\mathbf{a^m : a^n = a^{m-n}; a \in \mathbb{R}\backslash\{0\}; m \in \mathbb{Z}} \text{ and } \mathbf{n \in \mathbb{Z}}$$

Examples: $3^4 : 3^2 = 3^{4-2} = 3^2$ and $\left(\frac{2}{3}\right)^5 : \left(\frac{2}{3}\right)^2 = \left(\frac{2}{3}\right)^{5-2} = \left(\frac{2}{3}\right)^3$

The **quotient of powers with the same exponent** results in a power with the same exponent and with a base equal to the division of the bases.

$$a^m : b^m = (a : b)^m; a \in \mathbb{R}; b \in \mathbb{R}\backslash\{0\} \text{ and } m \in \mathbb{Z}$$

Examples: $3^2 : 4^2 = (3 : 4)^2 = \left(\frac{3}{4}\right)^2$ and $\left(\frac{3}{5}\right)^2 : \left(\frac{4}{5}\right)^2 = \left(\frac{3}{5} : \frac{4}{5}\right)^2 = \left(\frac{15}{20}\right)^2$

Power of power is a power with the same base whose exponent is equal to the product of the exponents.

$$(a^n)^m = a^{m \times n}; a \in \mathbb{R}; m \in \mathbb{Z} \text{ and } n \in \mathbb{Z}$$

Examples: $(4^3)^2 = 4^{3\times2} = 4^6$ and $\left(\left(\frac{1}{2}\right)^5\right)^2 = \left(\frac{1}{2}\right)^{5\times2} = \left(\frac{1}{2}\right)^{10}$

Power of base 0: $0^n = 0$ with $n \in \mathbb{N}$

Example: $0^4 = 0$

Power of exponent 0: $a^0 = 1$ with $a \in \mathbb{R}\backslash\{0\}$

Example: $5^0 = 1$

Power of base 1: $1^n = 1$ with $n \in \mathbb{Z}$

Example: $1^4 = 1$

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Power of exponent 1: $a^1 = a$ with $a \in \mathbb{R}\backslash\{0\}$

Example: $5^1 = 5$

Power of a negative exponent:

$$a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n} \text{ and } \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n;\ a \in \mathbb{R}\backslash\{0\}, b \in \mathbb{R}\backslash\{0\} \text{ and } n \in \mathbb{N}$$

Examples: $3^{-4} = \left(\frac{1}{3}\right)^4 = \frac{1}{3^4}$ and $\left(\frac{2}{3}\right)^{-2} = \left(\frac{3}{2}\right)^2 = \frac{3^2}{2^2}$

Power of a rational exponent:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m};\ a \in \mathbb{R}, m \in \mathbb{Z} \text{ e } n \in \mathbb{N}$$

Examples: $3^{\frac{5}{2}} = \sqrt[2]{3^5} = \sqrt{3^5}$ and $5^{\frac{-2}{3}} = \sqrt[3]{5^{-2}}$

Observations:

- The sign of a power with a positive base and a natural exponent is positive;
- The sign of a power with a negative base and a natural exponent is:
  - positive if the exponent is an even number;
  - negative if the exponent is an odd number.
- $(-a)^n = a^n$ if n is an even number.
- $(-a)^n = -a^n$ if n is an odd number;
- $(a^m)^n$ and a^{m^n} usually display different results because $(a^m)^n = (a^m) \times (a^m) \times \ldots \times (a^m)$ (n times) and $a^{m^n} = a^{m \times m \times \ldots \times m}$ (n times)

  Example: $(5^2)^3 = 5^{2\times3} = 5^6$ and $5^{2^3} = 5^8$

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TASKS

TASK 1

Calculate the value of:

1.1. $(-2)^3$ **1.2.** 1^5 **1.3.** $\left(\frac{2}{5}\right)^2$ **1.4.** 0^{10} **1.5.** $\left(-\frac{1}{2}\right)^4$

TASK 2

Fill in the gaps to obtain correct statements:

2.1. $4^3 \times 4^5 = 4^{\dots}$;

2.2. $(-3)^{\dots} \times (-3)^5 = (-3)^8$;

2.3. $5^7 : 5^5 = 5^{\dots}$;

2.4. $\left(-\frac{3}{2}\right)^{\dots} : \left(-\frac{3}{2}\right)^5 = \left(-\frac{3}{2}\right)^3$;

2.5. $(3^5)^{\dots} = 3^{10}$;

2.6. $(\dots^5)^3 = 3^{15}$;

2.7. $\left(\frac{27}{8}\right)^2 = \left(\frac{\dots}{\dots}\right)^6$

TASK 3

Turn the following powers into just one power with a positive exponent:

3.1. $4^{-3} \times \left(\frac{1}{2}\right)^{-3}$

3.2. $(3^{-2})^6 \times 5^{12}$

3.3. $\frac{5^{-8}}{5^3}$

3.4. $\left(\frac{3}{2}\right)^{12} : \left(\frac{2}{5}\right)^{-12}$

TASK 4

Calculate the numerical value of each of the following expressions, in accordance with the operation rules of powers:

4.1. $\left(\frac{5}{4}\right)^{-7} \times \left(2 + \frac{1}{2}\right)^7 : \left(\frac{1}{2}\right)^{-1}$;

4.2. $\left(\frac{1}{4}\right)^{-5} : 2^7 + (0,1)^{-1}$;

4.3. $\frac{\left(\frac{2}{5}\right)^{12} \times \left(\frac{2}{3}\right)^{-12}}{\left(1 - \frac{2}{5}\right)^{10}}$.

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LEARN MORE...

Chromatic Scale

https://www.youtube.com/watch?v=2gy6E3X2mKQ

Exponent Properties with products

https://www.khanacademy.org/math/pre-algebra/pre-algebra-exponents-radicals/pre-algebra-exponent-properties/v/exponent-properties-involving-products

Properties of Exponents

https://www.mathplanet.com/education/algebra-1/exponents-and-exponential-functions/properties-of-exponents

Laws of Exponents - NCERT Class 7th Maths Solutions

https://www.youtube.com/watch?v=9VuLwo1FBys

12-Tone Equal Temperament

http://www.tonalsoft.com/enc/number/12edo.aspx

Why 12 notes to the Octave?

https://www.math.uwaterloo.ca/~mrubinst/tuning/12.html

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