PART II: Music & Mathematics

AGE RANGE: 13 –15

"Sheet Music"
(Source: https://www.pexels.com/photo/black-and-white-keys-music-note-534283/)

TOOL 16: NUMERICAL SERIES IN HARMONIC SERIES

SPEL – Sociedade Promotora de Estabelecimentos de Ensino

TheArtOfMaths

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Educator's Guide

Title: Numerical Series in Harmonic Series

Age range: 13 – 15 years old

Duration: 3 hours

Mathematical concepts: Numerical series

Artistic concepts: Harmonic series in music, musical notes and frequencies of musical notes

General objectives: To understand numerical series' concept and be able to calculate some of its terms, as well as the n^{th} term

Instructions and Methodologies: In order for students to get a clearer picture of the mode of vibration, please have them watch "Modes on a string" video (cf. "Learn More…") after the respective explanation

Resources: A pen; Task 1 requires a paper box, an elastic and two crayons.

Tips for the educator: Begin by giving some examples of numerical series to explain its concept and then teach how to complete them. Explain how to calculate terms of numerical series by showing an example.

Learning Outcomes and Competences:

At the end of this tool, the student will be able to:

- Identify numerical series;
- Calculate terms of numerical series;
- Calculate the n^{th} term of numerical series

Debriefing and Evaluation:

Write 3 aspects you liked about this activity:	1. 2. 3.
Write 2 aspects that you have learned	1. 2.
Write 1 aspect for improvement	1.

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Introduction

Mathematics and Music have always been connected. However, the first evidence of this relationship was only found in the sixth century BC that the first evidence of this relationship were discovered. Pythagoras compared the sound produced by hammers of different lengths, used by blacksmiths, to the sound of the monochord, of which it is believed Pythagoras was the inventor.

This comparison allowed Pythagoras to discover and improve the mathematical reasons behind the sounds through the study of the sounds produced by the monochord. He divided the string into two equal parts, then into three equal parts, and so on. He matched the sounds mathematically according to the subdivisions he was making and created the Pythagorean scale, in which each note maintained a well-defined relationship with the other.

Many people and cultures have created their own scales. One example was the Chinese people who has created the pentatonic scale. Western culture, however, adopted a 12-tone equal temperament, known as a tempered scale or chromatic scale.

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Harmonic Series

It is of general knowledge that the natural musical notes are A, B, C, D, E, F and G. Nevertheless, these are represented in most countries by the solfège naming convention Do–Re–Mi–Fa–Sol–La–Ti (or Si) in accordance to the following correspondence: C-Do, D-Re, E-Mi, F-Fa, G-Sol, A-La and B-Ti (or Si). The definition of these notes was widely influenced by Mathematics.

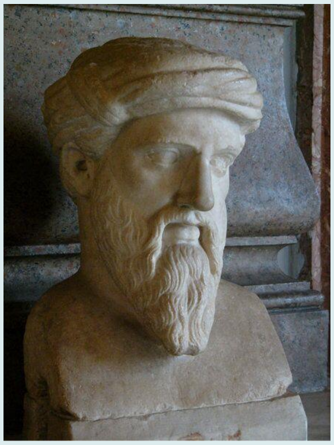

Fig. 1 – Pythagoras Bust
(Source:https://commons.wikimedia.org/wiki/File:Kapitolinischer_Pythagoras_adjusted.jpg)

In the sixth century BC, Pythagoras realized that when vibrating a string it not only vibrated in its full extent, but it also formed a series of nodes, which divide into smaller sections, the partials, which vibrate at frequencies higher than the fundamental.

To study the relationship between the length of the vibrating string and the musical tone produced by it, he used a monochord.

Figure 2 shows the nodes and partials of the first four frequencies of a series. For an easy understanding, they are shown separately, but on a real string, all overlap, generating a complex design, similar to the waveform of the instrument.

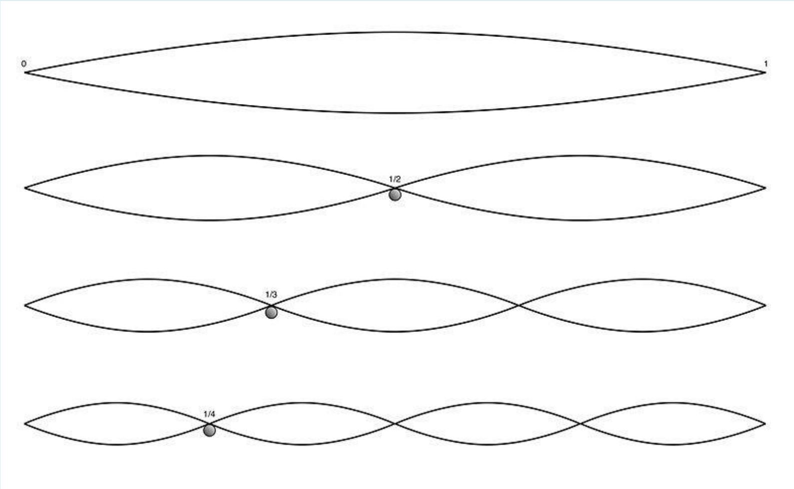

Fig. 2 – Modes of vibration of the first 4 harmonics
((Source:https://pt.wikipedia.org/wiki/Frequ%C3%AAncia_fundamental#/media/Ficheiro:Overtone.jpg)

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Imagine a string stretched out, stuck at its ends. When we touch one end of this string, it vibrates (note the first drawing in Figure 3) and produces a note that is called a fundamental note.

Fig. 3 – Modes of vibration of a fundamental note 1(f)
(Source:https://pt.wikipedia.org/wiki/Frequ%C3%AAncia_fundamental#/media/Ficheiro:Overtone.jpg)

Pythagoras decided to divide a string into two parts (Figure 4) by touching it in the middle. The sound produced was exactly the same, but with a higher frequency (usually expressed as "same note, an octave higher"). It has since been proved that whenever the number of divisions (or the harmonic number) is a multiple of an earlier number, then the sound will be repeated but with a higher pitch.

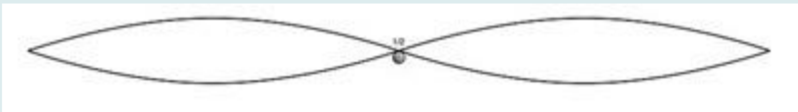

Fig. 4 – Modes of vibration of a fundamental note 2(f)
(Source:https://pt.wikipedia.org/wiki/Frequ%C3%AAncia_fundamental#/media/Ficheiro:Overtone.jpg)

He then decided to try out what it would sound like if the string was divided into 3 parts (Figure 5) and noticed that a new sound, different from the previous one, came out. This time, it was not the "same note, an octave higher", but a completely different note, which deserved a different name - the fifth.

Fig. 5 – Modes of vibration of a fundamental note 3(f)
(Source:https://pt.wikipedia.org/wiki/Frequ%C3%AAncia_fundamental#/media/Ficheiro:Overtone.jpg)

This sound, although different, matched well with the previous sound. It created a pleasant harmony to the ear, which had to do with the fact that the divisions done had the mathematical relations of 1/2 and 2/3. With the division of the string into four parts, he obtained the note, now known as "fourth". These three notes are in consonance with the fundamental note.

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Thus, he continued to subdivide the string, obtaining the harmonics of the fundamental note, and, by mathematically combining the sounds, he created scales which result in notes that naturally related to each other. Over time, the notes have been given the names we know of today, which were mentioned earlier.

In this process, each note coming from an object, suffers the influence of the fundamental frequency that excites other harmonics, which results in a series of frequencies - the harmonic series. The harmonic series are infinite series, composed of sinusoidal waves with all the integer multiple frequencies of the fundamental frequency. There is not a single harmonic series, but rather a different series for each fundamental frequency.

Let us look at an example of a harmonic series that starts at A_2 / $Lá_1$ (110 Hz). The first 16 harmonics for that series can be observed in the following table:

Harmonic #	Note (English)	Note (Neo-latin)	Frequency (Hz)
1 (F)	A_2	$Lá_1$	110
2	A_3	$Lá_2$	220
3	E_4	Mi_3	330
4	A_5	$Lá_3$	440
5	$C\#_5$	$Do\#_4$	550
6	E_4	Mi_4	660
7	G_4	Sol_4	770
8	A_5	$Lá_4$	880
9	B_5	Si_4	990
10	$C\#_6$	$Do\#_5$	1100
11	$D\#_6$	$Ré\#_5$	1210
12	E_6	Mi_5	1320
13	$F\#_6$	$Fá\#_5$	1430
14	G_6	Sol_5	1540
15	$G\#_5$	$Sol\#_5$	1650
16	A_6	$Lá_5$	1760

Table 1 – First 16 harmonics

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Glossary

Fifth: interval between a musical note and another, which is four degrees away from the first, within a scale.

Fourth: interval between one musical note and another, which is three degrees away from the first, within a scale.

Frequency: physical quantity indicating the number of occurrences of an event in a given time span.

Fundamental Frequency: the lowest and strongest component frequency of the harmonic series of a sound.

Fundamental note: main note of a chord, from which the other chords derive from

Harmonic Series: set of waves composed by the fundamental frequency and of all the integer multiples of this frequency.

Harmonic: sound of a series that constitutes a note.

Harmony: simultaneous combination of sounds.

Monochord: an old musical instrument composed of a resonance box, on which was extended a single string fastened by two mobile supports.

Octave: interval between a musical note and another one with half or twice its frequency.

Pentatonic scale: set of all scales consisting of five notes or tones.

Pitch: high frequency sound from human hearing, usually above 5 KHz.

(Musical) Scale: ordered sequence of tones by the vibratory frequency of sounds (usually from the lowest frequency sound to the highest frequency sound).

Tempered Scale: division of the octave into twelve equal semitones.

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Maths behind the Harmonic Series

The divisions made by Pythagoras on the string of the monochord correspond to divisions of the unit by natural numbers, i.e. follow the sequence 1, 2, 3, 4, ..., n. In other words, they correspond to the sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}$. In terms of vibration, the correspondence would be the sequence $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n-1}{n}$.

Numerical Sequences

A **sequence** or succession is a set of objects of any nature, organized or written in a specific order.

Examples:
The set January, February, March, April, ..., December is a sequence or succession of the months of the year.
The set 0, 1, 2, 3, 4, ... is called sequence or sequence of natural numbers.

A **numerical sequence** is a function that has as a domain the set of natural numbers and as a codomain (target set) the set of real numbers.

Numerical sequences can be finite when it is possible to "count" their elements, or infinite, when it is not possible to "count" their elements. See below the mathematical representations of both situations:

Finite sequence: $(a_1, a_2, a_3, \dots, a_n)$

Infinite Sequence: $(a_1, a_2, a_3, \dots, a_n, \dots)$

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Reading of the previous terms:

$a_1 \to a$ index 1 (first term or term of order 1)

$a_2 \to a$ index 2 (second term or term of order 2)

$a_3 \to a$ index 3 (third term or term of order 3)

$a_n \to a$ index n (n^{th} term or term of order n)

Take a look at the examples of finite and infinite sequences:

Finite sequence: (5, 7, 9, 11, 13, 15, 17, 19)

Infinite sequence: (3, 5, 7, 11, 13, 17, ...)

The sequences are mostly given by a mathematical formula called the **general term**. This is the law of formation that defines any of the terms of the sequence.

Examples:

1) Given the sequence defined by $a_n = 4n - 1$, with $n \in \mathbb{N}$, calculate the first and third terms.

Remember that the domain of this sequence is $\mathbb{N}$, so the first term is a_1 and is calculated by replacing $\mathbf{n}$ for 1.

For $\mathbf{n = 1}$, we have: $a_1 = \mathbf{4 \times 1 - 1 = 3}$

For $\mathbf{n = 3}$, we have: $a_3 = \mathbf{4 \times 3 - 1 = 11}$.

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1) Consider the arithmetic sequence 5, 8, 11 ...

The first term of the sequence is 5 and after the first term, any other term is obtained by adding 3 to the above term.

See, for example, the following calculations related to the first terms:

n	**n^{th} term calculation**	
1	5	$= 5 + 0 \times 3 = 5$
2	5 + 3	$= 5 + 1 \times 3 = 8$
3	5 + 3 + 3	$= 5 + 2 \times 3 = 11$
4	5 + 3 + 3 + 3	$= 5 + 3 \times 3 = 14$
5	5 + 3 + 3 + 3 +3	$= 5 + 4 \times 3 = 17$

Table 2 – Numerical sequence calculation

The table shows that we can obtain the term of order n (where n is any term order) from the first term 5 and by adding the ratio 3 repeatedly to $n-1$ times. This can be written algebraically as $\mathbf{5+3(n-1)}$. The simplified version of this expression has the form of $\mathbf{3n+2}$.

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TASKS

TASK 1

Using a paper box, an elastic and two crayons, reproduce the experiment conducted by Pythagoras with a monochord.

Note that by changing the position of one of the crayons, a different pitch is produced. Find out more in the following video:

https://www.youtube.com/watch?v=AQJw95-H9mM

TASK 2

Write the first four terms of the sequence (u_n), considering that:

2.1) $u_n = 5n - 2$

2.2) $u_n = -3n + 1$

2.3) $u_n = \frac{1}{n}$

2.4) $u_n = \frac{1}{3n}$

2.5) $u_n = \frac{1}{n^3}$

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TASKS 3

Write the following two terms and the general term of the sequence (u_n), where the first terms are:

3.1) $3, 6, 9, 12, 15, \ldots$

3.2) $4, 9, 14, 19, 24, \ldots$

3.3) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

3.4) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

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LEARN MORE…

The Maths of Music

https://www.youtube.com/watch?v=rTT1XHJKKug

Modes on a string

https://www.youtube.com/watch?v=cnH2ltfW48U

The Harmonic Series

https://www.oberton.org/en/overtone-singing/harmonic-series/

A path to understanding musical intervals, scales, tuning and timbre

http://in.music.sc.edu/fs/bain/atmi02/hs/hs.pdf

Pythagoras and Music

https://ba278b9d8106536501a2-57da1f3fe93ccf3a9828e6ce67c3d52c.ssl.cf5.rackcdn.com/07_richards.pdf

Numerical sequences

https://www.mathsisfun.com/numberpatterns.html

Musical Note to Frequency Conversion Chart

https://www.audiology.org/sites/default/files/ChasinConversionChart.pdf

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