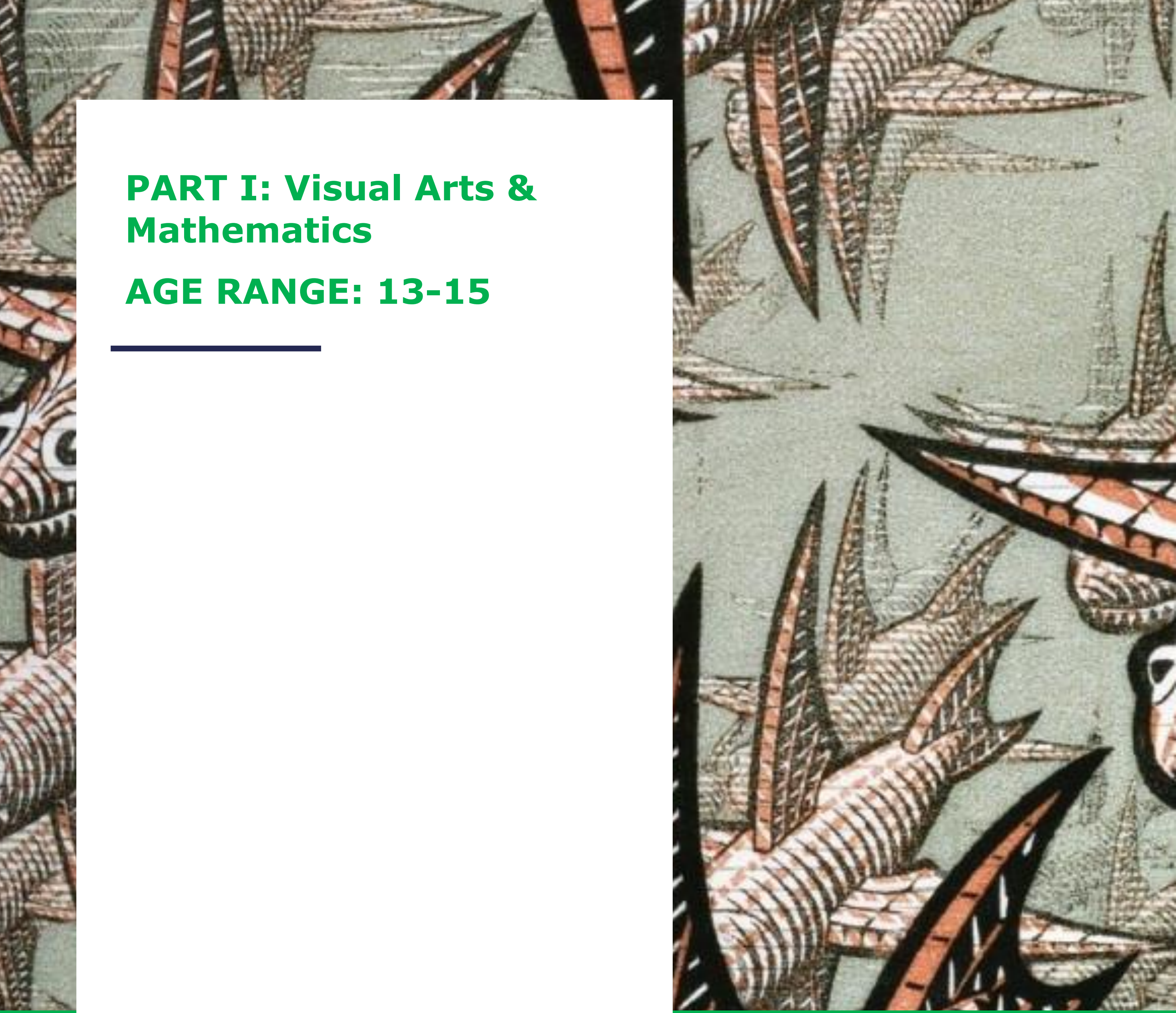

PART I: Visual Arts & Mathematics

AGE RANGE: 13-15

TOOL 6: THE MATHEMATICAL ART OF M.C. ESCHER

C.I.P. Citizens In Power

TheArtOf Maths

Co-funded by the Erasmus+ Programme of the European Union

Educator's Guide

Title: The Mathematical art of M.C. Escher; a retrospection of Escher's work from a mathematical point of view

Age Range: 13-15

Duration: 3 hours (Note: this is a double session, so it counts as a double tool)

Mathematical Concepts: regular division of a plane; symmetry; translation, reflection and rotation; Platonic Solids

Artistic Concepts: woodcut; linocut; majolica tiles; mosaics; Moorish tiles/tessellations; motif; lithograph

General Objectives:

-The students to comprehend the way in which the famous artist M.C. Escher has drawn from math theories and concepts in order to be capable of depicting -even artistically- the geometric illustrations which have characterized his work.

-The students to become able to comprehend the transition from *plane geometry* to *stereometrics*, thus addressing any difficulties might arise during the process of drawing solids in the plane, such as the cube, the parallelepiped, the prisms, the platonic solids.

Instructions and Methodologies: After discussing Escher's life mainly through his artworks, this tool can enhance the establishment of the conception and illustration of the plane, through the usage of two suggested tasks.

Tips for the educator: The educator can gradually start introducing in an intuitive way the basic math concepts while presenting Escher's artworks. Subsequently, the educator should explain to the students the mathematical theories and concepts analysed under 'The Math Behind' Section.

Resources: This tool provides pictures and videos on Escher's artworks. It also includes lively examples of the terminologies, references and some extra material.

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Learning Outcomes and Competences:

(i) Students will have comprehended the way in which Escher borrows from mathematical theory, in order to create his artistic creations, thus investigating common borders of art and science

(ii) Students will have gained a better understanding on how three-dimensional solids are being depicted in a plane, thus acquiring the ability to clearly perceive and represent the concept of three-dimensional space.

Debriefing and Questions for Evaluation:

As part of reflection and/or formative assessment (=in order to improve the tool for the next time according to the students' background, interest, exact age, country's culture, students' prior knowledge etc) the educator can use these cards sometimes called EXIT CARDS either by a hard copy he/she has made from before or simply by posing these statements on board and the students write the answers on a paper which they will leave preferably anonymously while exiting the room. The specific formative strategy is called 3,2,1. For more strategies you can visit: https://www.bhamcityschools.org/cms/lib/AL01001646/Centricity/Domain/131/70%20Formative%20Assessments.pdf

3-2-1	
Write 3 things you liked about this activity	1. 2. 3.
Write 2 things you have learned	1. 2.
Write 1 aspect for improvement	1.

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Introduction

Even though the mathematical perspective of Dutch artist M.C. Escher is widely known, only a small minority of those who have been admiring him have indulged their passion on the mathematical side of his creations. Escher had inevitably interacted with mathematicians by first comprehending and subsequently borrowing mathematical forms, with the aim to incorporate them in his artworks. He mainly employed geometric representations, as no other artist did after Renaissance, approaching even abstract mathematical concepts through visual metaphors. Escher insisted on the depiction of infinity after conducting his own mathematical research on correlated theories, therefore establishing channels of both communication and conciliation with widely recognised mathematicians, such as Pólya, Penrose, and Coxeter, whilst the latter have many times been led to further theoretical developments and discoveries, inspired by Escher's alternative way of approaching mathematical objects.

Escher's Mathematical Artworks

M.C. Escher was born in Leeuwarden (1898) and grew up in Arnhem, Holland. He came from a family full of scientists; his father was a civil engineer, whilst his four siblings had all scientific backgrounds. As Schattschneider describes in his research work about Escher, "the home atmosphere may have instilled in him some habits of scientific inquiry, including the patient, methodical approach that would characterize his later work. Also, the young boys [Escher and his brothers] were given regular lessons in woodworking techniques that would later become very useful to Escher in making woodcuts.

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Picture 1: [Skull], 1919 or 1920; from the collection of M.C. Escher's woodcuts and Wood Engravings, Early works, 1916-1921[Retrieved from [http://www.eschersite.com/EscherSite/Basalt_Rocks_Escher_31.html]

His school life may have been less useful than his home life. Recalling his school years, Escher once confessed "I was an extremely poor pupil in arithmetic and algebra, and I still have great difficulty with the abstractions of figures and letters. I was slightly better at solid geometry because it appealed to my imagination, but even in that subject I never excelled at school" […]."He did well in drawing, however, and his high school art teacher encouraged him to make linocuts" (Schattschneider, 2010, p. 706).

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Picture 2: Head of a Child, Lino Cut in green, 1916, 89 mm x 114mm

Picture 3: The Rag Pickers, linoleum cut in black, wine-red,purple,1918,160mmx200 mm

Picture 4: Parrot, linoleum cut, 1919, 166mm x 275mm

(**Pictures 2-4:** Retrieved from: https://www.mcescher.com/gallery/)

M.C. Escher got accepted to study Architecture at the Haarlem school of Architecture and Decorative Arts; however, it was his teacher, Samuel Jessurun de Mesquita, who soon encouraged him to switch to a graphic design course. After graduating, in 1922, Escher travelled in Italy and Spain, where he focused on the depiction of landscapes, buildings, as well as on some parts of the nature. During his trip Alhambra (Granada, Spain), Escher was inspired by the geometric aesthetics decorating majolica tiles, and particularly by the way of constructing the tiling in this area, thus being led to the design of his own collections.

Accordingly, in 1920s and as Schattschneider describes, M.C. Escher "produced a few periodic "mosaics" with a single shape, some of them hand-printed on silk. Unlike the Moorish tiles that always had geometric shapes, Escher's tile shapes (which he called "motifs") had to be recognizable (in outline) as creatures, even if only of the imagination. These early attempts show that he understood (intuitively, at least) how to utilize basic congruence-preserving transformations— translations, half-turns (180°

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rotations), reflections and glide-reflections—to produce his tilings" (Schattschneider, 2010, p.707).

Picture 5: Moorish tessellations of Alhambra inspired Escher's work with tilings of the plane.

After getting married in 1924, he continued to make trips in the Southern Italy with the aim to employ some design elements in his drafts for lithographs and woodcuts, while in1936 he decided to move with his family to Switzerland, due to the rise of Italian fascism. In one of his family trips in Alhambra, Escher sketched the majolica tiling of Alhambra.

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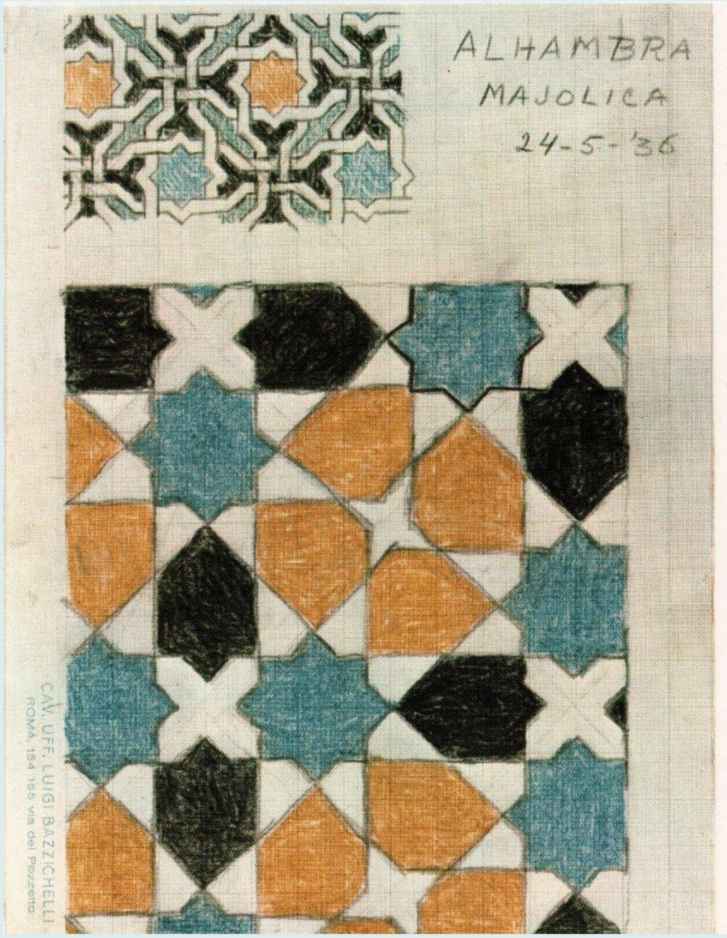

Picture 6: Sketches that Escher made in 1936, as he was influenced by majolica tilling of Alhambra. He used pencil and coloured crayon

(Retrieved from: http://pi.math.cornell.edu/~mec/Winter2009/Mihai/section8.html)

This was the second time in which Escher had visited Alhambra whilst this moment could be considered as a turning point for his work, in terms of both his style and the thematic areas from which he had used to draw inspiration. This fact could be attributed to the replacement of the sketched 'landscapes', resulted from various influences from nature, animals and architecture by the so-called "mindscapes", thus focusing more on the geometrical relationship of the tiles with one another, as well as on ways of creating symmetrical drawings of interlocked motifs.

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Picture 7-11: Watercolours of Escher that display the notion of 'symmetry' through symmetrical drawings of interlocked motifs (1938-1950)

(**Pictures 7-11**: Retrieved from: https://www.mcescher.com/gallery/symmetry/)

In other words, Escher began to encapsulate the inventions of his own imagination, whilst simultaneously experiencing his inner need for further study of the scientifically proven theories of mathematics, geology and crystallography. As it seems the Mauritanian geometric designs, which for religious reasons had a complete absence of every human form, attracted him unimaginably; theoretically such designs could continue to 'infinity'.

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Accordingly, from this period, Escher's illustrations mainly incorporate a geometric design; he made usage of contradictions, while his engravings used to approach an 'infinite time and space'.

Symmetry and Platonic Solids

Escher mostly worked with symmetries, rings and spheres in space, reflected images, inversions, rotations, relativities, as well as with the collision between the plane and the space.

He frequently embedded three-dimensional objects, the so-called Platonic solids, such as spheres, cubes and tetrahedrons in his artworks. He also used cylinders and stellated polyhedral, whilst in many times he used to make combinations of two-dimensional shapes with three-dimensional shapes, focusing on the concept of dimensionality. Accordingly, his work sparked the interest of well-reputed mathematicians, such as for instance Doris Schattschneider and Roger Penrose, who both focused on geometric distortions.

For instance, in his "Waterfall" (1961) we can observe two compound polyhedra, namely a compound of three cubes and a stellated rhombic dodecahedron, both located on the top of an impossible building (Picture 6). The latter (rhombic dodecahedron) is widely known as Escher's solid, which had also been used in his work "Stars" (1948) in which all five of Platonic solids are being incorporated, in combination with other stellated solids which represent stars (Picture 6).

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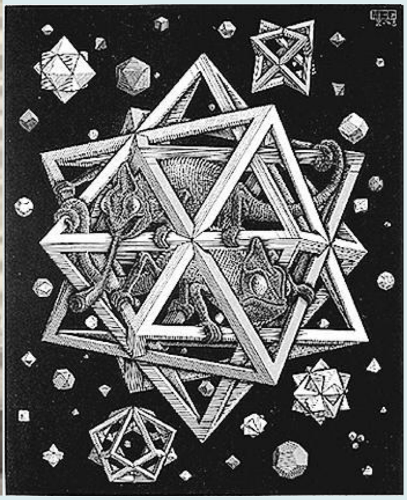
Picture 12: Waterfall, 1961 Lithograph



Picture 13: Stars, 1948, wood engraving

(**Pictures 12-13**: Retrieved from: https://www.mcescher.com/gallery/impossible-constructions/)

Below you can see artworks deriving from the mathematical collection of Escher, in which he had employed stereometric figures, such as rings and spheres, through a collision between the plane and the space.

Picture 14: 'Stars', 1948, wood engravings in colour

Picture 15: 'Concentric Rinds, 1953, wood engraving

Picture 16: Sphere Spirals, 1958, Woodcuts in grey, black, yellow and pink, printed from 4 blocks

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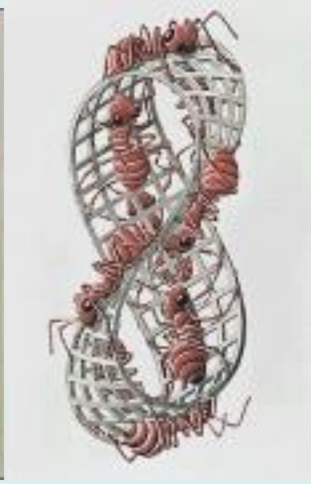

Picture 17: Sphere surface with fish, 1958, woodcut in grey, gold and reddish brown, printed from 3 blocks

Picture 18: Mobius Strip II, 1963, woodcut in red,

Picture 19: Knots, 1965, woodcut in red, green and brown, printed from 3 blocks

(**Pictures 14-19**: Retrieved from: https://www.mcescher.com/gallery/mathematical/)

The "Regular Division of a Plane" within Mathematical Patterns

However, Escher has become famous for the regular division of plane along with his so-called tessellations. A tessellation of a flat surface is the tiling of a **plane** using one or more **geometric shapes**, called tiles, with no overlaps and no gaps.

It seems that Escher was heavily influenced by the definition of the mathematician Haag of 'regular division of a plane' given as follows:

> "Regular divisions of the plane consist of congruent convex polygons joined together; the arrangement by which the polygons adjoin each other is the same throughout"

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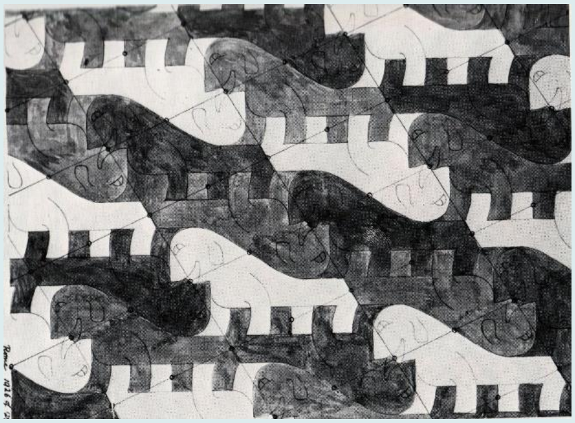
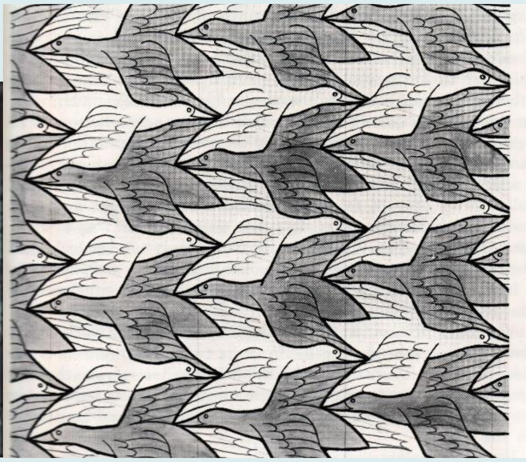


Picture 20: Escher's first attempt (1926 or 1927) to maintain a regular division of a plane, using imaginary animals (Retrieved from: http://pi.math.cornell.edu/~mec/Winter2009/Mihai/section8.html)

Creating "Metamorphosis" through the usage of tessellations

In this section we are going to discuss the process of "metamorphosis" an integral part of Escher's creations, and the way in which he used to employ the so called "tessellations" with the aim to maintain a "metamorphosis". As a first step, it's being noted that Escher used "tessellations" as an intermediary step, and not as a main theme in his artworks. Metamorphoses were achieved through the gradual transition from abstract figures and shapes to strictly delineated concrete forms, whilst the latter were turning into something else. Some examples of the intermediary phase, namely the "tessellation phase" along with the final product are being provided below:



Picture21: In the left-hand side, the famous woodcut of Escher, namely 'Day and Night' (1938),

Picture22: the tessellation that Escher used in order to create this woodcut.

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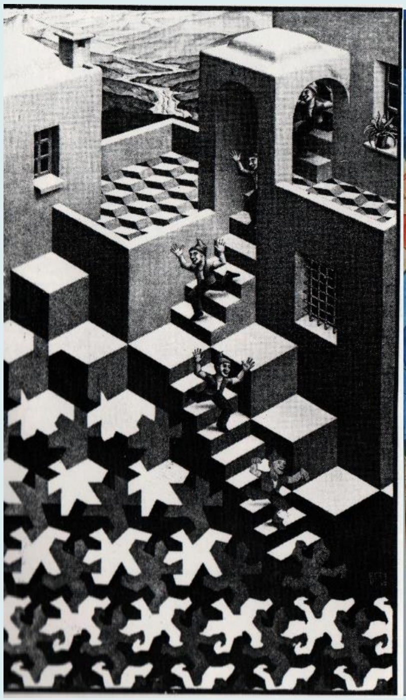
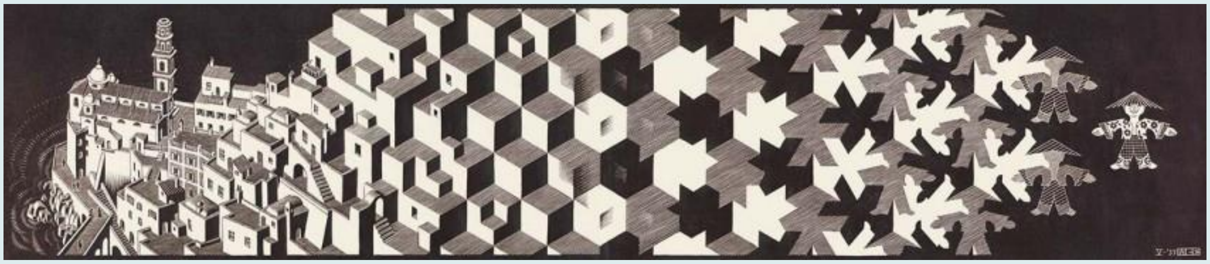



Picture 23: In the left-hand side, Escher's lithograph 'Cycle' of 1938,

Picture 24: the periodic space filling (tessellations) which constitute the basis for reaching 'Cycle' final result.

For instance, his work 'Metamorphosis I' (1937) represents three buildings in Atrani coastal town transformed into cubes which eventually unfolded into the Chinese boys.



Picture 25: Metamorphosis I, 1937, woodcut printed in two sheets (Retrieved from: https://www.mcescher.com/gallery/switzerland-belgium/metamorphosis-i/)

Within his book "Plane Tessellations" (1958) Escher provides further explanations regarding the stages he goes through in order to create a "metamorphosis".

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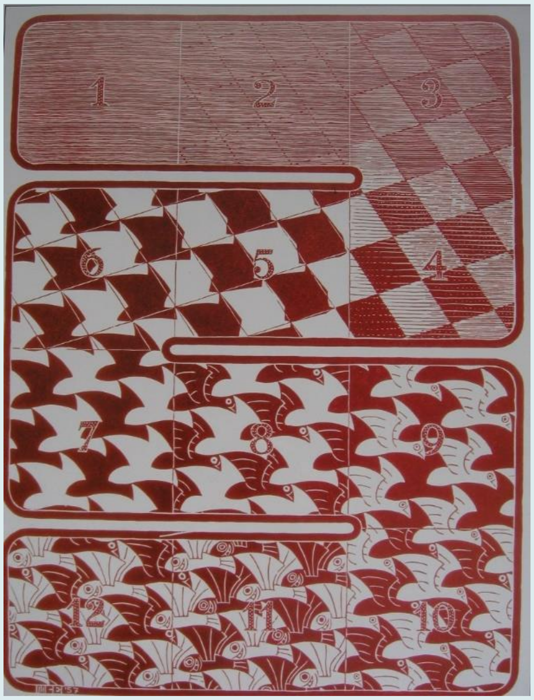

Picture 26: The stages of a metamorphosis (Retrieved from: http://pi.math.cornell.edu/~mec/Winter2009/Mihai/section8.html)

As one could observe in the picture above, from phase 1 to phase 4 the plane is being divided into black and white parallelograms. In the next phase (5) the shape of the parallelograms is gradually being changed, in a way that an outward bulge on one of the sides results in an equal inward bulge in the opposite side. The parallelograms continue to alter, mainly in terms of size, whilst much more details are gradually added, thus reaching phase 8, in which all the black shapes begin to look like birds. Subsequently (phase 9-10) the same happens with the white shapes, which also turn into birds. Although, in phases 11-12, through the addition of some more details, the birds are being converted to fish (Online Source: http://pi.math.cornell.edu/~mec/Winter2009/Mihai/section8.html).

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GLOSSARY

Woodcut: a print of a type made from a design cut in a block of wood, formerly widely used for illustrations in books.

Linocut: Linocut is a printmaking technique, a variant of woodcut in which a sheet of linoleum (sometimes mounted on a wooden block) is used for a relief surface. A design is cut into the linoleum surface with a sharp knife, V-shaped chisel or gouge, with the raised (uncarved) areas representing a reversal (mirror image) of the parts to show printed. The linoleum sheet is inked with a roller (called a brayer), and then impressed onto paper or fabric. The actual printing can be done by hand or with a printing press.

Moorish Architecture: Moorish architecture is the articulated Islamic architecture of North Africa and parts of Spain and Portugal (Al Andalus), where the Andalusians (Moors) were dominant between 711 and 1492. The best surviving examples in Iberia are La Mezquitain Córdoba and the Alhambra palace in Granada (mainly 1338–1390), as well as the Giralda in Seville (1184). Other notable examples in Iberia include the ruined palace city of Medina Azahara (936–1010), the church (former mosque) San Cristo de la Luz in Toledo, the Aljafería in Saragossa and baths at for example Ronda and Alhama de Granada.

Tessellations: A tessellation of a flat surface is the tiling of a plane using one or more geometric shapes, called tiles, with no overlaps and no gaps. In mathematics, tessellations can be generalized to higher dimensions and a variety of geometries.

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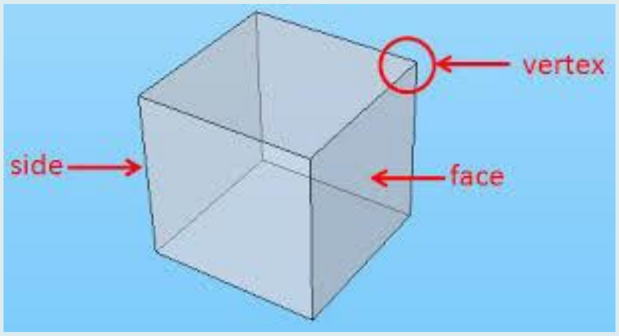
(I) Platonic Solids

In the following picture 27 you can see a cube, which constitutes one of the five Platonic Solids, along with its side, face and vertex, where in simple words:

- **Face** is a single flat surface of a solid
- **Side or edge** is a line segment between faces
- **Vertex** is a corner



Picture 27: Cube, one of the five Platonic Solids

The following Table represents the five Platonic Solids, namely tetrahedron, octahedron, icosahedron, cube, dodecahedron. In the first column you can see the way of depicting each one of the solids; in the second column the number of faces of each solid are being given; in the third and fourth column, the number of Edges and Vertices respectively of each solid; in the fifth column the formula of Surface Area as well as of Volume of each solid is being provided, given that a is the length of the side.

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the five Platonic solids

Name	Faces	Edges	Vertices	A = Surface Area V = Volume a = length of side
 tetrahedron	4 equilateral triangles	6	4 3 faces meeting	$A = \sqrt{3}\,a^2$ $V = \frac{\sqrt{2}}{12}a^3$
 octahedron	8 equilateral triangles	12	6 4 faces meeting	$A = 2\sqrt{3}\,a^2$ $V = \frac{\sqrt{2}}{3}a^3$
 icosahedron	20 equilateral triangles	30	12 5 faces meeting	$A = 5\sqrt{3}\,a^2$ $V = \frac{5(3+\sqrt{5})}{12}a^3$
 cube	6 squares	12	8 3 faces meeting	$A = 6a^2$ $V = a^3$
 dodecahedron	12 regular pentagons	30	20 3 faces meeting	$A = 3\sqrt{25+10\sqrt{5}}\,a^2$ $V = \frac{15+7\sqrt{5}}{4}a^3$

Picture 28: The Platonic Solids (Retrieved from: http://www.amathsdictionaryforkids.com/qr/p/PlatonicSolids.html)

(II) TRANSFORMATION THEORY

- **Translation** is widely known as "slide". It slides all the points of a shape in the same distance by following the same direction. Translation does not have any effect on the shape of the figure. For instance, in the triangle QRP each point has been transferred 3 units downwards (y axis) and at the same time 4 units right. The new triangle that arises is the Q'R'P'.

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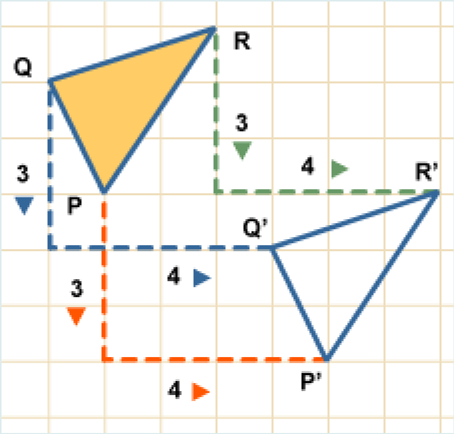
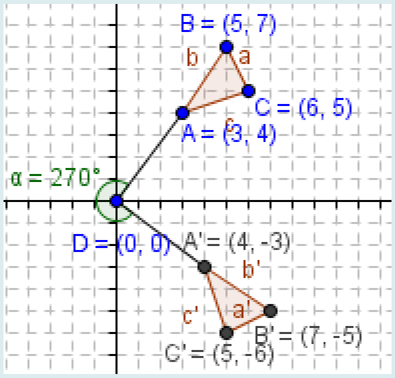


Picture 29: Translation

- **Rotation** is widely known as a "turn". The rotated shape can move upward, downward, right, left in the plane. However, it should always turn around a certain point which is known as the centre of rotation. Rotation does not have any effect on the shape's size.

  For instance, the triangle ABC makes a rotation of 270 degrees around the point D(0,0) which is the centre of location.



Picture 30: Rotation

- **Reflection** is widely known as a "Flip". It could be described as the mirror image of the shape, therefore changing all the points over a line. It merely reflects the figure over a line, dubbed as "line of reflection".

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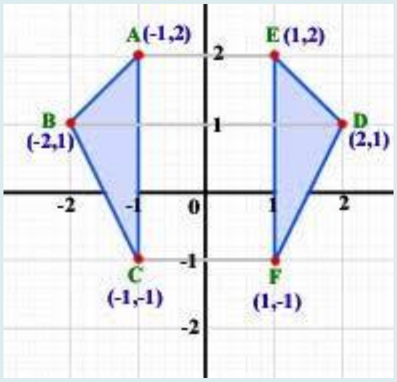
For instance, in the following picture yy' axis could be considered as the line of reflection of the triangle ABC, while the triangle EDF constitutes the reflected image (mirror image) of ABC.



Picture 31: Reflection

Principles of Plane Tessellation

In this paragraph we will attempt to explain the mathematical concepts of translation, rotation and reflection through the "artistic eyes" of Escher. Let's say that we have a surface, fully covered with equilateral triangles, according to Escher's Regular division of a plane.

As described in http://pi.math.cornell.edu/~mec/Winter2009/Mihai/section8.html "if we shift the whole plane over the distance AB, it will cover the underlying pattern once again. This is a translation of the plane. We can also turn the duplicate through 60 degrees about the point C, and we notice that again it covers the original pattern exactly. This is a rotation. Also if we do a reflection about the line PQ, the pattern remains the same. A pattern can be made to map to itself by means of translation, rotation, reflection and glide reflection. There are 17 different groups of patters. Each groups admits only some kinds of shifts whereby they map onto themselves (some admit only translation, others translation and reflection etc.). Escher discovered all these possibilities without any previous mathematical knowledge. A particular characteristic of Escher's tessellation is that he chooses motifs that represent concrete

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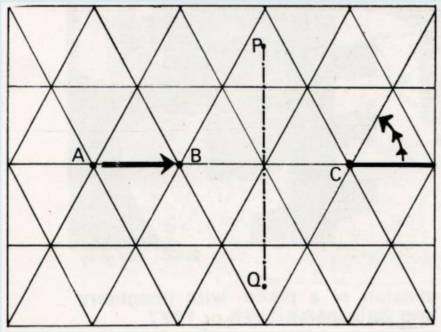
objects or beings" (Online source:

http://pi.math.cornell.edu/~mec/Winter2009/Mihai/section8.html)



Picture 33: (Retrieved from: http://pi.math.cornell.edu/~mec/Winter2009/Mihai/section8.html)

Searching the Net, with the aim to retrieve information about the symbolization of trefoil in gothic temples (Retrieved from: https://study.com/academy/lesson/trefoil-in-architecture-definition-design.html)

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TASKS

TASK 1

Transforming a plane into a Platonic solid

As we have observed, Escher focuses on the transformation of polygons, by following the rules of the "Regular Division of a Plane". He attempts to give a sense of three-dimensional constructions within the plane, by mostly working with contrasts, such as black-white, day-night. He makes use of different tones of grey in order to achieve the transition from one to the other.

Escher commented on his work 'Waterfall': If we watch the different parts of this construction one by one, we cannot find any error in it. And yet it is a total impossibility because changes occur suddenly when interpreting the distance between the eye and the object ".

This dual interpretation of "what really exists" and "what we see" is not just a "game" which could be detected in Arts. It often constitutes the heart of a mathematical proof.

A typical example of what is being implied here, constitutes the problem of calissons (French sweets of a roughly rhomboid shape), which was first set by Hallenbeck, Deturck and Solow, in 1989. According to this, we randomly place the sweets in a hexagonal box, so that there are no gaps in it (see picture below). We will soon notice that the sweets could be categorised -according to their orientation in the plane- into three groups (located north to south, north-west to south-east, north-east to south-west).

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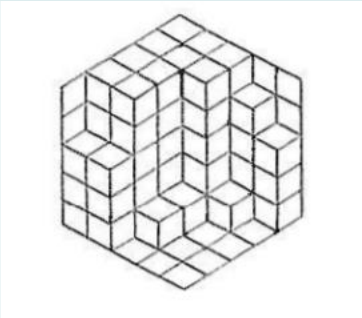
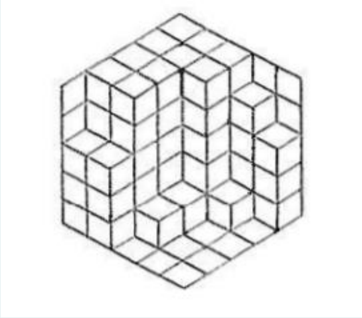


Picture 34: The calissons problem: Filling a hexagonal plane with rhomboid calissons (sweets)

Question 1: What do you observe regarding the number of sweets of each of the three categories?

Question 2: By using 3 different tonalities (for instance: black, grey and white), colour in the following picture the sweets of each category with the same colour.



Question 3: After completing the previous task given within Question 2, what do you observe regarding the hexagonal plane? To which Platonic solid is it being transformed, and which is the role of rhomboid calissons in this case?

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TASK 2

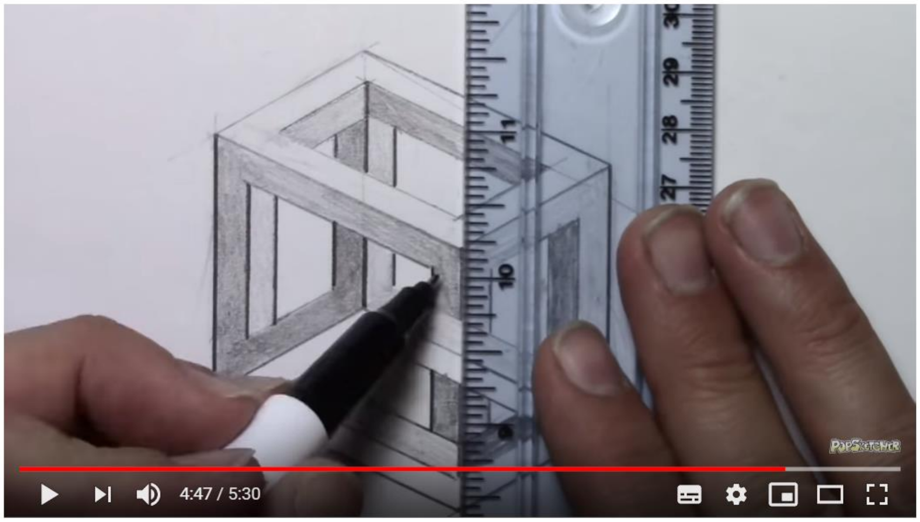
Escher's impossible cube

Watch the following YouTube videos:

https://www.youtube.com/watch?v=CZAQ_b2rzAA

https://www.youtube.com/watch?v=OM9oYcdIJaM

Subsequently, by following the guidelines contained therein, try to draw **Escher's impossible cube.**



How-to-draw an Optical Illusion - Escher Cube

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LEARN MORE…

If you want to further investigate on the topics addressed in this tool, you may go through the following links:

M.C. Escher; The Official Website:
https://www.mcescher.com/

The Mathematical Side of M.C. Escher by Doris Schattschneider:
https://www.ams.org/notices/201006/rtx100600706p.pdf

M.C. Escher's: More Mathematics Than Meets the Eye:
http://www.msri.org/people/members/sara/articles/siamescher.pdf

The Art of the Impossible: MC Escher and Me- Secret Knowledge:
https://www.youtube.com/watch?v=f7kW8xd8p4s

The Mathematical Art of M.C. Escher:
http://platonicrealms.com/minitexts/Mathematical-Art-Of-M-C-Escher/

Escher's Tessellations of a Plane:
http://pi.math.cornell.edu/~mec/Winter2009/Mihai/section8.html

How to Draw an Optical Illusion- Escher Cube:
https://www.youtube.com/watch?v=CZAQ_b2rzAA

https://www.youtube.com/watch?v=OM9oYcdIJaM

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