PART II: Music & Mathematics

AGE RANGE: 16-18

TOOL 19: RATIOS OF FREQUENCIES OF MUSICAL NOTES

C.I.P. Citizens In Power

Co-funded by the Erasmus+ Programme of the European Union

Educator's Guide

Title: Ratios of frequencies of musical notes

Age Range: 16-18

Duration: 2 hours

Mathematical Concepts: Intervals, Frequencies, Ratios, logarithms.

Artistic Concepts: Math behind making music.

General Objectives: To show that adding intervals is equal to multiplying frequencies and a method for describing any interval or its ratio in terms of an octave, perfect fifth and major third, a concrete numerical value for tuning ratios (a definition of cents).

Instructions and Methodologies: The tool is based on a general introduction on the relationship between maths and music but then it becomes a bit more challenging in the Math Behind Section. It was attempted to make it as light and understandable as possible by images, examples, pictures and a youtube video.

Resources: Youtube Videos, Books, Journals, Pictures and Glossary

Tips for the educator: You can start by some general questions to intrigue their interest on how and if they believe that maths and music are related. Then you can have a quick read of the introduction before going to the Math behind the Music.

Desirable Outcomes and Competences: Realize that intervals and the relationships between notes can be reduced to combinations of the first three overtones

Debriefing and Evaluation Questions

3-2-1	
Write 3 things you liked about this activity	**1.** **2.** **3.**
Write 2 things you have learned	**1.** **2.**
Write 1 aspect for improvement	**1.**

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Introduction

Music and math are related to each other and, in fact, it is said that mathematics can help us explain the musical experience. Grandin, Peterson, and Shaw (1998) state that music improves reasoning skills, which are crucial to learning mathematical concepts such as proportional reasoning and developing geometry skills. Rauscher et al. (1997) claim that music promotes the development of such thinking skills and especially recognizing patterns and using logic.

Pythagoras the Ancient Greek, even from the 6th century BC, realized that different sounds can be made with different weights and vibrations. This led to his invention that the pitch of a vibrating string is proportional to and can be controlled by its length. Strings that are halved in length are one octave higher than the original. Thus, the shorter the string, the higher the pitch. Pythagoras also found out that notes of certain frequencies sound best with multiple frequencies of that note. For example, a note of 220Hz sounds best with notes of 440Hz, 660Hz, and the like. So you can already see that from the very basics to the most complicated synthesis, maths is intertwined with music.

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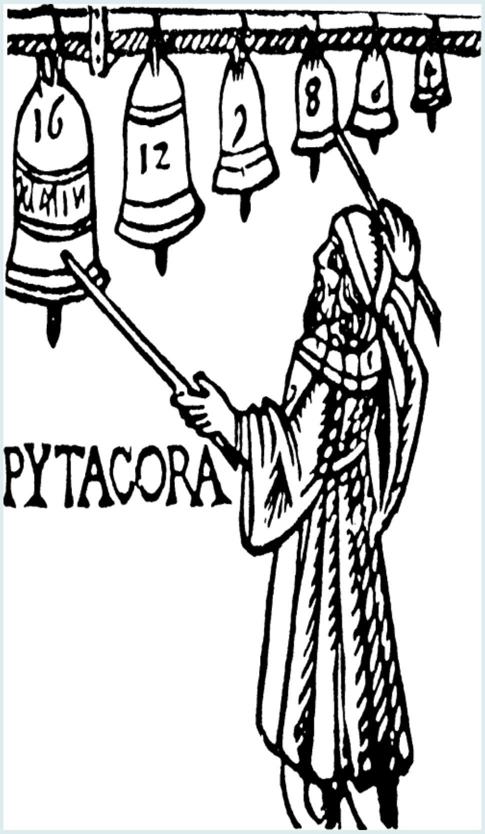

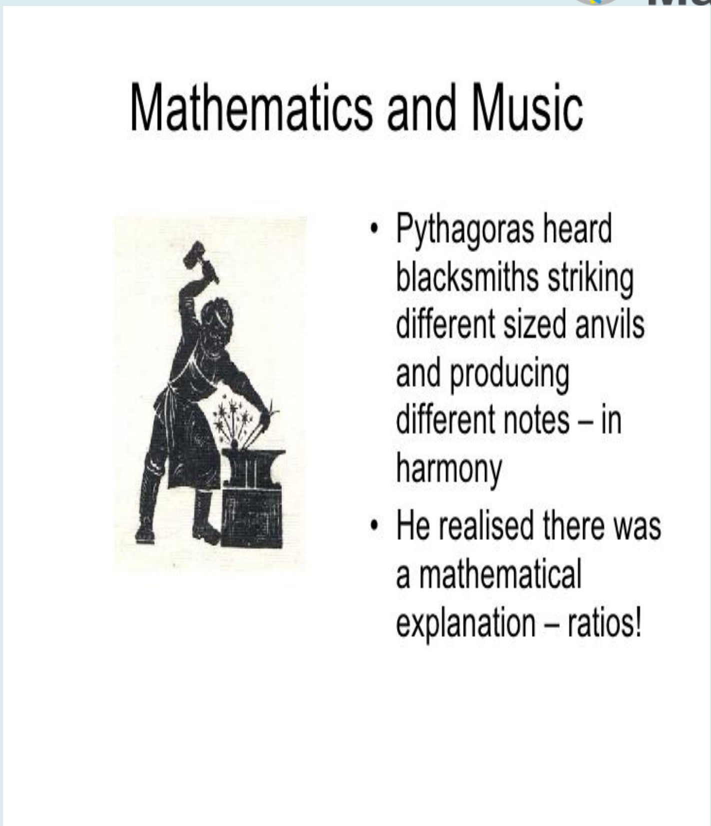

Picture 1: Diatonic Scale with the Concept of Pythagoras (Retrieved from**:** https://www.google.com/search?q=pythagoras+and+music&client=firefox-b-d&source=lnms&tbm=isch&sa=X&ved=0ahUKEwjN9r7B_IPjAhXJCuwKHcEKD6AQ_AUIECgB&biw=1138&bih=527#imgrc=pAHlvMTRAjeGWM)

Picture 2: Pythagoras and Music (Retrieved from: https://www.google.com/search?q=pythagoras+and+music&client=firefox-b-d&source=lnms&tbm=isch&sa=X&ved=0ahUKEwjN9r7B_IPjAhXJCuwKHcEKD6AQ_AUIECgB&biw=1138&bih=527#imgrc=kjndmRILmsTNLM)

Frequencies, intervals and ratios in music

Some common mathematical concepts that relate to music are namely, the frequencies, scales, intervals and tones.

Pythagoras made his discoveries by "playing" with a stretched string. Below you can see a stretched string tied in its extremities. When touched, it vibrates. As we all know when an instrument is vibrating, a wave of pressure that travels through the air reaches our ear as a sound.

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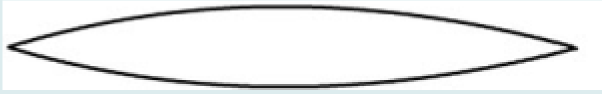
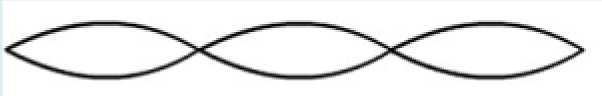


Pythagoras decided to divide this string in two parts and touched each extremity again. The sound that was produced was the same, but more acute (because it was the same note one octave above):

Pythagoras decided to continue. He experimented with the string divided in 3 parts:



That is when he realized that a new, different sound appeared. This time, it wasn't the same note one octave above, but a different note, that had to receive another name. This sound, besides being different, worked well with the previous one, creating a pleasant harmony to the ear, because these divisions showed the Mathematics relations 1/2 and 2/3 and apparently our brain likes fit defined logic relations.

Thus, he continued doing subdivisions and combinations of the sounds mathematically creating scales that, afterward, stimulated the creation of musical instruments that could play these scales. Nowadays, the notes received the names we know today. Cultures have created their own scales. For example, the Chinese created the pentatonic scale, whilst Western culture, adopted a 12-tone equal temperament, known as a tempered scale or chromatic scale.

Sources: http://www.simplifyingtheory.com/mathematics-and-music/ and http://mathcentral.uregina.ca/beyond/articles/Music/music1.html

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Glossary

Pythagoras: Pythagoras of Samos[a] (c. 570 – c. 495 BC) was an ancient Ionian Greek philosopher and the eponymous founder of Pythagoreanism. His political and religious teachings were well known in Magna Graecia and influenced the philosophies of Plato, Aristotle, and, through them, Western philosophy. Knowledge of his life is clouded by legend, but he appears to have been the son of Mnesarchus, a seal engraver on the island of Samos. Modern scholars disagree regarding Pythagoras's education and influences, but they do agree that, around 530 BC, he travelled to Croton, where he founded a school in which initiates were sworn to secrecy and lived a communal, ascetic lifestyle. This lifestyle entailed a number of dietary prohibitions, traditionally said to have included vegetarianism, although modern scholars doubt that he ever advocated for complete vegetarianism.

Retrieved from: https://en.wikipedia.org/wiki/Pythagoras

Ratio: In mathematics, a ratio is a relationship between two numbers indicating how many times the first number contains the second.[1] For example, if a bowl of fruit contains eight oranges and six lemons, then the ratio of oranges to lemons is eight to six (that is, 8:6, which is equivalent to the ratio 4:3). Similarly, the ratio of lemons to oranges is 6:8 (or 3:4) and the ratio of oranges to the total amount of fruit is 8:14 (or 4:7).

Retrieved from: https://en.wikipedia.org/wiki/Ratio

Frequency: Frequency is the number of occurrences of a repeating event per unit of time. The period is the duration of time of one cycle in a repeating event, so the period is the reciprocal of the frequency.[2] For example: if a newborn baby's heart beats at a frequency of 120 times a minute, its period—the time interval between beats—is half a second (60 seconds divided by 120 beats). Frequency is an important parameter used in science and engineering to specify the rate of oscillatory and vibratory phenomena, such as mechanical vibrations, audio signals (sound), radio waves, and light.

Retrieved from: https://en.wikipedia.org/wiki/Frequency

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Glossary

Octave: In music, an octave (Latin: octavus: eighth) or perfect octave (sometimes called the **diapason**) is the interval between one musical pitch and another with double its frequency. The octave relationship is a natural phenomenon that has been referred to as the "basic miracle of music", the use of which is "common in most musical systems". The interval between the first and second harmonics of the harmonic series is an octave. In music notation, notes separated by an octave (or multiple octaves) have the same letter name and are of the same pitch class. To emphasize that it is one of the perfect intervals (including unison, perfect fourth, and perfect fifth), the octave is designated P8. Other interval qualities are also possible, though rare. The octave above or below an indicated note is sometimes abbreviated 8a or 8va (Italian: all'ottava), 8va bassa (Italian: all'ottava bassa, sometimes also 8vb), or simply 8 for the octave in the direction indicated by placing this mark above or below the staff.

Retrieved from: https://en.wikipedia.org/wiki/Octave

Interval (music): In music theory, an interval is the difference in pitch between two sounds. An interval may be described as horizontal, linear, or melodic if it refers to successively sounding tones, such as two adjacent pitches in a melody, and vertical or harmonic if it pertains to simultaneously sounding tones, such as in a chord. In Western music, intervals are most commonly differences between notes of a diatonic scale. The smallest of these intervals is a semitone. Intervals smaller than a semitone are called microtones. They can be formed using the notes of various kinds of non-diatonic scales. Some of the very smallest ones are called commas, and describe small discrepancies, observed in some tuning systems, between enharmonically equivalent notes such as C♯ and D♭. Intervals can be arbitrarily small, and even imperceptible to the human ear.

Retrieved from: https://en.wikipedia.org/wiki/Interval_(music)

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Minor third: In the music theory of Western culture, a minor third is a musical interval that encompasses three half steps, or semitones.

Major third: In classical music from Western culture, a third is a musical interval encompassing three staff positions (see Interval number for more details), and the major third is a third spanning four semitones. Along with the minor third, the major third is one of two commonly occurring thirds. It is qualified as major because it is the larger of the two: the major third spans four semitones, the minor third three.

Fourth: interval between one musical note and another, which is three degrees away from the first, within a scale.

Perfect fifth: interval between a musical note and another, which is four degrees away from the first, within a scale.

Frequency: physical quantity indicating the number of occurrences of an event in a given time span.

Pitch: high frequency sound from human hearing, usually above 5 KHz.

Math behind… notes

- ✓ The sound is made from a continuous *vibration* of air.
- ✓ The number of vibrations per second is called frequency which is measured in *Hertz* and the *frequency* of sound determines its *pitch* (see glossary) <the higher the frequency, the higher the pitch>.
- ✓ Musical notes are sounds of certain frequencies playing an ascending order of frequencies which will produce a musical scale.
- ✓ Here is the explanation for the difference between two pitches. Consider two pitches (frequencies) which are separated by an arbitrary distance, i.

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Consequently if we have our two frequencies, f1 and f2, they are separated by the interval distance i1. (Interval is the combination of two of these sounds).

- ✓ Now the ratio of the two frequencies (f2 / f1) can be defined as r1 and can be expressed as:

$$f2 \div f1 \ = r1$$

- ✓ If we have a second set of frequencies, f3 and f4, the interval between them can be defined as i2. The ratio of f3 and f4 (being f4/f3) would be defined as r2. If i1 and i2 are the same interval, meaning that the same frequency distance exists between f1 and f2 as well as f3 and f4, then the ratios will be equal. This does not tell us anything about the register of the frequencies, only that the intervals are similar (we don't know if they're in the same register). We would express this as:

$$f2 \div f1 = r1$$

$$f4 \div f3 = r2$$

$$i1 \ \cong i2 \ \text{if and only if} \ r1 = r2$$

- ✓ If we have three frequencies f1, f2 and f3. The interval between f1 and f2 is i1, the interval between f2 and f3 is i2 and the larger interval between f1 and f3 is i3. Applying the same concepts of calculated ratios in the previous examples we get:

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$$f2 \div f1 = r1$$

$$f3 \div f2 = r2$$

$$f3 \div f1 = r3$$

$$\therefore f2 = r1 \cdot f1 \text{ and } f3 = r2 \cdot f2$$

$$Substituting\ in\ for\ f2$$

$$f3 = r2 \cdot (r1 \cdot f1)$$

$$f3 \div f1 = r2 \cdot r1$$

$$substituting\ r3\ for\ f3 \div f1$$

$$r3 = r2 \cdot r1 \text{ and } i3 = i1 + i2$$

- ✓ Therefore, we show that *adding intervals* is equal to *multiplying frequency ratios*.

$$since\ r3 = r2 \cdot r1$$

$$\log(r3) = \log(r2) + \log(r1)$$

$$since\ i3 = i2 + i1\ we\ can\ show\ that$$

$$i3 = \log(r3) \text{ and } i2 = \log(r2) \text{ and } i1 = \log(r1)$$

- ✓ Now we have a defined number for the value of i. *It is the log of the ratio of the frequencies comprising the interval in question*. The frequency ratio for any given interval will be positive, but it may be greater than or less than 1. If the value of r is greater than 1, then we know that $0 < f1 < f2$ and the interval is *ascending* (because f2 is greater than f1). Likewise if $0 < r < 1$ then $0 < f2 < f1$ and we know the interval is *descending*. Therefore the log of an ascending interval (with $r > 1$) will be *positive* while the log of a descending interval (with $r < 1$) will be *negative*.
- ✓ In piano playing DO and RE together is described as a major second interval because RE is the second note in the scale -> next is a major third interval because MI is the third note in the scale -> from DO to FA it is called the perfect

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fourth interval -> from DO to SOL which is the fifth note is called the perfect fifth interval and so on -> finally a DO and DO played together is called an octave (Now watch https://www.youtube.com/watch?v=rTT1XHJKKug until minute 2:08).

- ✓ If we know how to determine ratio of an interval formed from other ratios, for example, if we knew one interval (r1) had a ratio of 5/4 (called *major* third) and another (r2) the ratio 6/5 (a *minor* third) we can calculate the ratio of their sum. So a *major* third (5/4) plus a *minor* third (6/5) gives:

$$r1 \cdot r2 = r3$$

$$\left(\frac{5}{4}\right) \cdot \left(\frac{6}{5}\right) = \left(\frac{3}{2}\right)$$

- ✓ The ratio 3/2 is a perfect fifth. Thus, we mathematically proved from a *bare concept* that a major third plus a minor third gives a perfect fifth! A quick refresher for your small integer overtone pitch ratios so you can try some other examples on your own:

2/1	Octave
3/2	Perfect Fifth
4/3	Perfect Fourth
5/3	Major Sixth
5/4	Major Third
6/5	Minor Third

- ✓ We've taken the un-quantified concept of an interval, derived a real number value for it from the ratio of frequencies, and used our formula to calculate the ratio of a resultant interval. In the next weeks we'll derive formulas to calculate frequencies, define what a note is, determine a method for finding the composition of any interval, and show that scale are modular functions (and how to use them).
- ✓ All intervals can be described as different combinations of the octave, perfect fifth and major third – the first three overtones.

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- ✓ Accordingly:

  - o interval ratios are always greater than 1 and less than 2;
  - o A ratio of 2:1 is an octave; all the other intervals are smaller than an octave.

- ✓ Following the overtone series we can draw the following conclusions:

  - o 1:1 is our starting tone, 2:1 is an octave above that, 3:1 is a fifth above the octave, 4:1 is the second octave and 5:1 is a third in the second octave.

- ✓ We will now attempt to find a way to write fifth and third as frequencies within the first octave. We can take for granted that the ratio must be greater than 1:1 and less than 2:1 which is how the octave is being expressed.
- ✓ Therefore, we can divide the frequency ratio by the number of octaves necessary, thus managing to reach the first octave range.

  - o For instance, 3:1 ratio constitutes a perfect fifth, falling in the second octave. Consequently, we should only bring it down by one octave in order to get a ratio among 1 and 2. This could be expressed as follows:

    Let's express with the letter q/1 an interval where q should be always greater than 2 and at the same time a prime number. In order to achieve an octave reduction we have to find a number n which obeys to the following formula:

$$1 < \frac{q}{2^n} < 2$$

$$\Rightarrow \quad 2^n < q < 2 \cdot 2^n$$

$$\Rightarrow \quad 2^n < q < 2^{n+1}$$

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- ✓ We have now showed that a prime ratio can be transposed into a single octave.
- ✓ The **first three basic intervals** resulting from the application of the formula are: 2:1 octave, 3:2 perfect fifth and 5:4 major third.

- ✓ Do you remember that the interval length is given by the logarithm of its ratio? For instance, we can use variables to express the length of the first three basic intervals:

$$a=\log(2)$$

$$b=\log(3/2)$$

$$c=\log(5/4)$$

- ✓ Now that we have expressed variables a, b and c as the length of the interval (which can be estimated through the logarithm of the ratio of frequencies), we will subsequently work through the following MATHEMATICAL TASKS in order to prove a method to determine the composition of any other interval as a combination of those three.

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TASK

Main question: Prove that the any interval could be determined as a combination of the intervals a, b and c defined within the previous paragraph.

TIP 1: As an initial step, remember that we have already defined the length of the three basic intervals as

$a= \log (2)$
$b= \log (3/2)$
$c=\log (5/4)$

TIP 2: To begin with, use three new variables, namely, m, n and s to multiple against our originals (a, b and c). Consider that, the sum of these will be the length of the unknown interval we're trying to determine;

So, the length of the new interval can be expressed as:

$ma + nb + sc$

Auxiliary Questions:

By answering those intermediary questions, you will be able to answer to the main question of this task ("Prove that the any interval could be determined as a combination of the intervals a, b and c defined within the previous paragraph")

(a) Considering that the length of this new interval is ma+nb+sc, how could we define its ratio?

(b) Based on your findings of the previous questions, could you estimate the sum of a perfect fifth and a major third?
Tip: You can determine a perfect fifth with the variable b and and the major third with the variable c.

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LEARN MORE…

The Maths of Music Video:

https://www.youtube.com/watch?v=rTT1XHJKKug

TED TALK: Music and Math: The genius of Beethoven

https://www.youtube.com/watch?v=zAxT0mRGuoY

Webpages:

Math central: http://mathcentral.uregina.ca/beyond/articles/Music/music1.html

Kent State Univeristy: https://musicedmasters.kent.edu/the-connection-between-music-and-mathematics/

Mathematics and Music: http://www.simplifyingtheory.com/mathematics-and-music/

Math and Music Lessons: https://www.notreble.com/buzz/2010/02/04/math-and-music-intervals/

Books:

Grandin, T., Peterson, M., & Shaw, G. L. (1998). Spatial- temporal versus language-analytic reasoning: The role of music training. Arts Education Policy Review, 99(6), 11-15.

Kung, D. (2013). How Music and Mathematics Relate. The Great Courses, Virginia. Retrieved from http://www.chrysalis-foundation.org/1373_MusicandMath_8-28.pdf

Rauscher,R.H.,Shaw,G.I.,Levine,I. J.,Wright,E.L.,Dennis, W. R., & Newcomb, R. I. (1997). Music training causes long-term enhancement of preschool children’s spatial-temporal reasoning. Neurological Research, 19, 2-8.

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